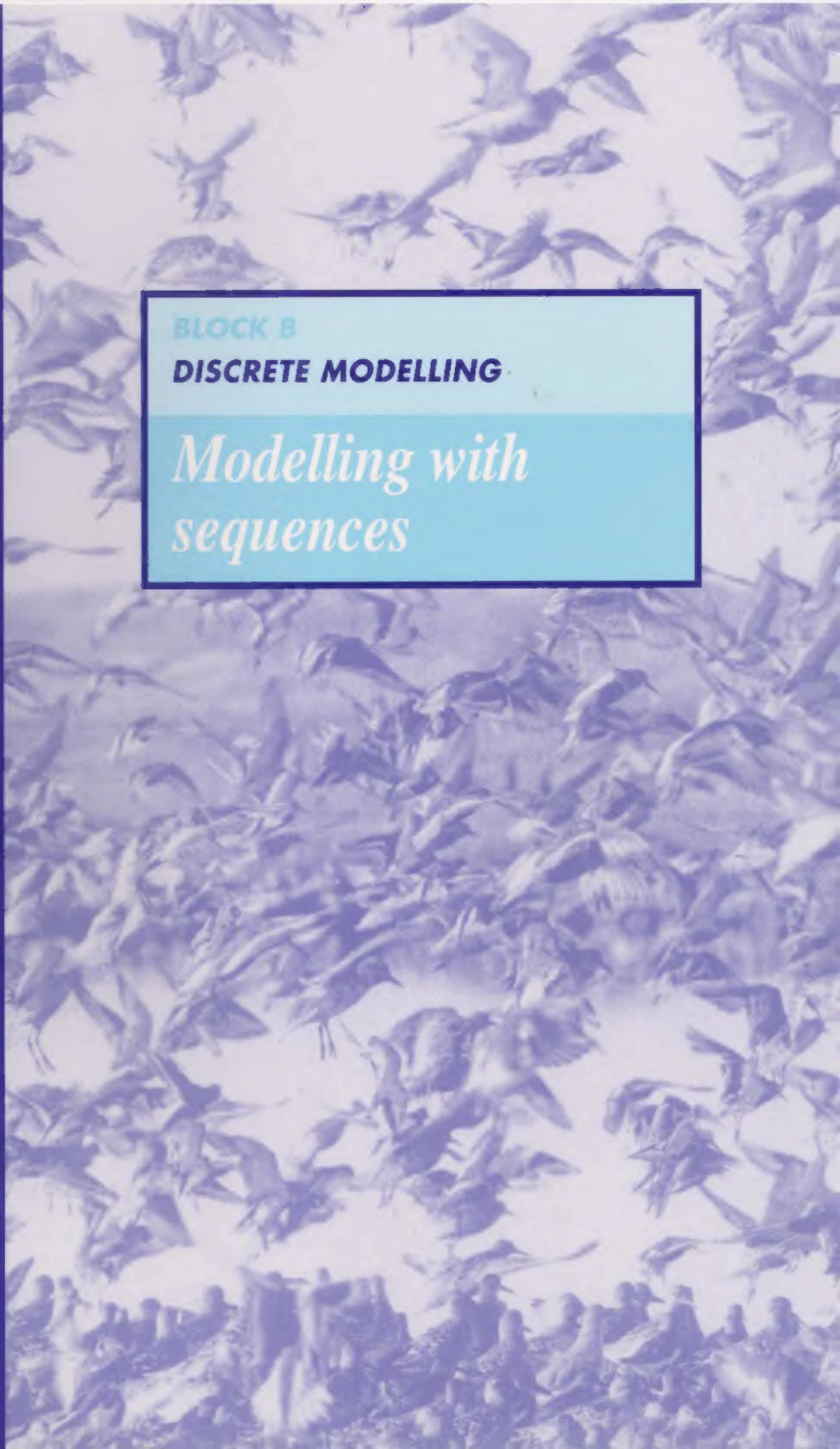
MST121 Chapter B1

The Open University

A first level interdisciplinary course

Using Mathematics



BLOCK B

DISCRETE MODELLING

Modelling with sequences

CHAPTER

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Prepared by the course team

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Contents

Introduction to Block B

Block B develops the theme of *mathematical modelling*, which was introduced in Block A. As in Chapter A1, the first two chapters of Block B are concerned with *discrete models*, that is, models in which the independent variable takes separate or discrete values (as opposed to values which range continuously over an interval of real numbers). For example, Chapters B1 and B2 are both concerned with models for *populations*, in which estimates for the population size are sought at regular intervals (typically once a year).

Chapter B1 looks at two successive population models, of which the second is a development of the first. This is in line with a general principle of modelling, that one should start with a simple representation of the real situation and then, if necessary, develop the model further. Chapter B1 permits you to revise to some extent what you learnt in Chapter A1 about solving first-order linear recurrence systems. It also examines what is meant by the *convergence* of a sequence to a *limit*, and introduces the topic of *series* (sums of the terms of a sequence).

Chapter B2 develops the initial population model of Chapter B1 in a different direction, by showing how to reflect the internal structure of a population. For example, members of a human population can be categorised as either juveniles or adults, and the variation in the sizes of these two subpopulations is a matter of interest to planners. This is modelled by using two dependent variables, one for each of the subpopulation sizes, rather than the single one used in Chapter B1. Each subpopulation size in a given year will depend significantly on *both* of the subpopulation sizes in the previous year, leading to a more complex type of recurrence system. Correspondingly, new mathematical entities are required to describe the model and to obtain predictions from it. The pair of subpopulation sizes is represented by a *vector*, and the array of coefficients for the recurrence relation makes up a *matrix*. There is a natural multiplication operation on matrices which is useful in this context and also in many others. In terms of matrices, a closed-form solution of the recurrence system can be found to describe the sequence of population vectors.

Chapter B3 takes a step aside from the theme of discrete models, to pursue further the topic of vectors. The algebraic approach to them in Chapter B2 can be related to a geometric view, in which vectors are seen as quantities which have *both magnitude and direction*. Thus wind velocity is one example of a vector quantity. The ways in which vectors can be combined make them a useful tool in many modelling situations. Often this involves an application of trigonometry, which is developed and practised further in the chapter. One important use for vectors is as a model for the *forces* which cause objects to move or, when in balance, keep objects from moving. The chapter and the block conclude with an introduction to the subject of *statics*, which concerns the relationships between forces which act on objects at rest.

Study guide

There are five sections in this chapter. They are intended to be studied consecutively in five study sessions. Each session requires two to three hours.

The pattern of study for each session might be as follows.

Study session 1: Section 1.
Study session 2: Section 2.
Study session 3: Section 3.
Study session 4: Section 4.
Study session 5: Section 5.

Section 4 requires the use of the computer together with Computer Book B.

Section 5 is independent of Section 4, so it can be studied earlier if you wish. Subsection 5.4 will not be assessed. Subsections 2.3 and 3.3 are just for reading: they contain no activities.

Introduction

You saw in Chapter A1 that sequences can be used in a variety of contexts. Here, we look in some depth at models based on sequences in two particular situations. The first, discussed in Section 1, relates to personal economics: 'How long should you keep a car before selling it?' The second, discussed in Sections 2–4, concerns the modelling of the number of individuals in some animal population, which is relevant to conservation, for example. Your work on these specific situations is intended to develop skills that you will need when creating and using models of your own. These skills include choosing variables, setting up a mathematical model, investigating the implications of the model, and interpreting the results of your calculations in terms of the original question that the model was intended to address.

While these are important skills, you are asked for the most part to appreciate their application rather than to master them at this stage.

In creating sequence models, it often proves natural to specify the sequence through a recurrence system. For some sequences given by recurrence systems, we can also describe the sequence by a closed form. Where such a formula can be found, it may be possible to use it to answer the original question. But even where such a formula cannot be found, the computer enables sequences given by recurrence systems to be calculated. In the model concerning car ownership, and in the first population model, we are able to find closed forms. This enables us to work to a greater extent algebraically (rather than numerically). In the second model for populations, there is no known closed form. In that case, you are invited (in Section 4) to use the computer to evaluate the sequences generated by the recurrence system in the model, and are asked to take note of the strategy used in carrying out such an investigation. You will see also that, even in the absence of a closed form, it is sometimes possible to use reasoning in order to cut down the scope of a numerical investigation.

In Sections 2–4 we pay particular attention to what happens to population sequences in the long term. In Section 5 we step aside from specific contexts, to look from a mathematical viewpoint at the question of how sequences behave. In particular, we look at sequences whose values 'settle down' close to a particular number in the long term. For example, the values of the sequence given by

$$a_n = 2 + 10^{-n} \quad (n = 1, 2, 3, \ldots)$$

are

$$2.1,\ 2.01,\ 2.001,\ 2.0001,\ \ldots,$$

and these are 'settling down' close to 2. Here, the number 2 is referred to as the *limit* of the sequence.

The concept of a limit is important in explaining the ideas of calculus, which you will meet in Block C.

1 Modelling car ownership

Being able to make use of mathematics to solve problems is an important skill for a mathematician to have. This skill is not an easy one to acquire, partly because there are no set procedures to follow. Even after you have studied this section, you are not expected to be able to create a model for yourself for some new problem that is unrelated to any that you have seen before. The aim here is to focus on the processes involved when you are in such a situation. Only a little new mathematics is introduced in this section, because the model constructed provides an application of the types of sequences that you studied in Chapter A1. Apart from appreciating the process of modelling, therefore, you will be able to use this section to some extent as revision of the work that you did in Chapter A1. You will also see how to use a new notation to describe the *sum* of terms of a sequence.

You will find that, in this course, your mathematical modelling skills are assessed in only minor ways. You should not, therefore, worry about the modelling process to the detriment of your study of the mathematics involved.

In creating mathematical models, there are a number of non-mathematical factors that need to be considered. For example, what features of the real situation should you attempt to incorporate into the model? Exactly what problem should you attempt to address through your model? Having created the model, is it adequate for its intended purpose? In this section you will see consideration of some of these wider issues in the context of a model concerning car ownership.

In Chapter A1 you met a five-stage diagram of the modelling process (see Figure 1.1). This diagram is a reminder of the main processes involved in modelling, and their logical order. However, it would be misleading to suggest that modelling consists of five clear-cut stages, performed in a fixed order. For example, when you start to create a model, you may find that you have set an unrealistic target in defining its purpose. You may then need to go back to the earlier stage, and refine your definition of 'the purpose'.

Chapter A1, Section 7

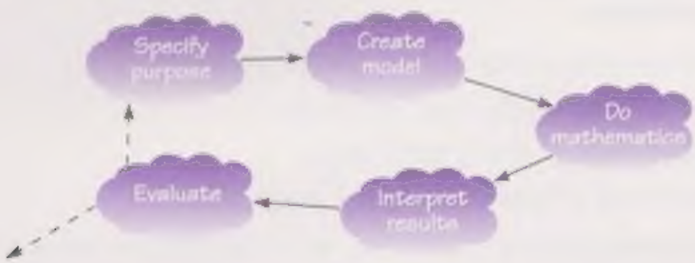


Figure 1.1 The modelling cycle

When you evaluate a model, you need to decide whether or not it is adequate for its purpose. If it is not, then you need to decide *why* this is the case. In doing so, you may examine the assumptions that you made in creating your model. Does the process of evaluation suggest ways in which you might modify these assumptions? If so, creation of a revised model, and repetition of Stages 2–4 in the modelling cycle, may lead to a more satisfactory outcome. In practice, several revisions may be needed. In the current section, however, we shall forego any revision to the model first presented.

1.1 Constructing a model

Many car owners have wondered whether it is better to buy a car new or used, and for how long to keep it. In this subsection, we construct an initial model to investigate some aspects of this question. This is probably not 'the best model that can be thought up', and the description given here will, in the interests of clarity and cohesion, omit the false starts and preliminary thoughts which occurred along the way. As the model is developed, you may well think of alternative approaches!

Specify the purpose

There are many factors that people might consider important in deciding which type of car to buy. Among these are passenger and luggage capacity, fuel consumption, emission levels and engine size. To simplify matters, let us suppose that the choice of car make and model has already been made.

Activity 1.1 Listing features

Do not spend more than a minute or two on this activity.

Assume that you have decided which type (make and model) of car you wish to buy, but not its age nor how long you will keep it. Note down some features that you might take into account in deciding your strategy for car purchase.

A solution is given on page 55.

Specify purpose

Before creating a mathematical model, the purpose for which it is intended must be clarified. It is sensible here to concentrate on features that lend themselves to mathematical description. This suggests that we should stick to aspects of the situation that can be measured; hence aspects such as the colour and condition of the vehicle should be discounted for modelling purposes.

Items which are measurable include the following:

- age and mileage at which car is bought;
- age and mileage at which car is sold;
- cost of the car at purchase, and its resale value;
- running costs, such as repairs and servicing, fuel, insurance and tax.

Although this list of features is restricted to measurable quantities, it is still too complex to cope with as it stands. We need to pick out the item that is considered to be of most importance, and then to identify what other quantity or quantities it mainly depends upon. An item of considerable interest to many is the cost of motoring, so it seems natural here to concentrate upon that. Thus we aim to construct a model which will show how to *keep the cost of motoring as low as possible*, in a sense yet to be made precise.

How can this be achieved? If cost is to be the output, or *dependent* variable, within the model, what inputs or *independent* variables does it depend upon? If the values of the inputs can be altered as we please, then it may be possible to select them in order to achieve some 'lowest cost' for the motorist.

For a first model, it is appropriate to simplify as far as possible. Let us therefore leave out of consideration the mileage of the car, and concentrate instead on its age as the factor which determines its value at any time. Various 'lowest cost' questions can be posed, for example the following.

This is equivalent to assuming that the car always registers a 'normal mileage' for its age.

- If the car is bought at three years old, at what age should it be sold?
- If it is to be sold at ten years old, at what age should it be bought?
- If it is to be kept for just two years, at what age should it be bought?

We shall fix on the following question.

> Suppose that a car of given type is purchased when it is one year old. At what age should it be sold in order to minimise the cost of owning and running the car?

While this statement of purpose is more precise than 'keep the cost of motoring as low as possible', we have not yet fully tied down what the 'cost of owning and running the car' involves. Each year, the car will depreciate in value, which is to say that what you can expect to receive for it at the end of the year will be less than what it was worth at the beginning of that year. This decline in value is an annual 'cost' which may vary from year to year. The running costs for the vehicle will also recur annually but on a varying basis. The annual ownership and running cost is the sum of these two items. We seek to minimise this cost *as averaged over the length of time between purchase and resale of the car.*

This average cost is a measure which is more appropriate to the consumer than the total cost of owning and running the vehicle. For example, a car which is kept for 5 years and costs £10 000 overall works out at an average of £2000 per year. Another car which is kept for 10 years and costs £15 000 in all (more in total than the first car) averages only £1500 per year, which is more economical than the first car.

We are now able to state the purpose of the model precisely, as follows.

Purpose of model

A car of given type is purchased when it is one year old. At what age should it be sold in order to minimise the average annual cost of owning and running the car?

Create the model

Now that the purpose of the model has been tied down, we define appropriate variables to correspond to the major features identified above.

- i (in years): age of the car ($i = 0$ when the car is new).
- n (in years): length of time for which the car is owned (purchased at one year old, resold at age $n + 1$ years).

Note that i and n are integers.

- a_n (in £): average annual cost of owning and running the car for n years.
- v_i (in £): value of the car at age i years, where $i = 0, 1, 2, \ldots, n + 1$ (what it costs to buy the car, or what its resale yields).
- c_i (in £): annual cost of servicing and repairs, from the beginning of year i to the beginning of year $i + 1$, where $i = 0, 1, 2, \ldots, n$.

As you will see, v_0 and c_0 *are* relevant in the construction of the model.

In the above list, n is the independent variable (the input which can be varied at will) and a_n is the dependent variable (the output, which is to be minimised). The other variables, i, v_i and c_i, are needed as intermediate quantities, permitting us eventually to express the output, a_n, in terms of the input, n.

Note that c_i includes only the running costs of servicing and repairs, and not items such as fuel, insurance and tax. For a given type of car with normal mileage, these latter items are assumed to be the same each year, so they will not affect the purpose stated for our model.

At this point we need to make assumptions about how both the value of the car and its servicing and repair costs vary with age. The depreciation in the value of any article with time is usually dealt with in one of two ways:

Sometimes a combination of these two approaches is adopted.

- by assuming that the article has a definite fixed life, and that its value declines to zero linearly over that life;
- by assuming that the value of the article declines by a fixed proportion each year.

Where used cars are concerned, the second approach seems to be closer to actual valuations than the first. This observation does *not* extend to the initial year of a car's life, because of the substantial additional premium which is paid to acquire a brand-new vehicle. However, since we are considering a car bought at one year old, depreciation by a fixed proportion will fit the bill here. Hence we have

$$v_{i+1} = rv_i \quad (i = 0, 1, \ldots, n),$$

Data are introduced here only for illustrative purposes. They would not normally appear this early in the modelling cycle.

where r is a constant such that $0 < r < 1$. Data gathered for one particular type of car, for illustrative purposes within this model, give $r = 0.85$ and $v_0 = 16\,000$. This value of r is equivalent to a depreciation rate of 15% per annum. The value of v_0 can be regarded as the initial cost of the car minus the 'brand-new' premium referred to above (or as what you might expect to receive if you bought a new car and then immediately tried to resell it).

Hence, for this type of car, v_i is given by

$$v_0 = 16\,000, \quad v_{i+1} = 0.85v_i \quad (i = 0, 1, \ldots, n). \tag{1.1}$$

Reliable data on the costs of servicing and repairs are less easy to come by. It is generally acknowledged that these annual costs of running a car increase with the age of the vehicle, and the simplest approach consistent with this observation is to assume that these costs increase linearly with age; that is,

This is consistent with the *principle of simplicity* within mathematical modelling, which states that, in the absence of any other guide as to how to model a relationship between two variables, you should choose the simplest form of function which takes into account the known facts

$$c_{i+1} = c_i + d \quad (i = 0, 1, \ldots, n - 1),$$

where d is a positive constant. Some (admittedly rough and ready) data for the type of car being considered here lead to the values $c_0 = 600$ and $d = 250$. Hence c_i is given by

$$c_0 = 600, \quad c_{i+1} = c_i + 250 \quad (i = 0, 1, \ldots, n - 1). \tag{1.2}$$

Equations (1.1) and (1.2) will permit v_i and c_i to be expressed in terms of car age, i years. How can we then relate the dependent variable of the model, a_n, to n, v_i and c_i?

The net loss involved in buying the car when one year old and selling it n years later is $v_1 - v_{n+1}$. Also, the total cost of running the car between purchase and sale is

$$c_1 + c_2 + \cdots + c_n.$$

The total cost of owning and running the car for n years is therefore

$$v_1 - v_{n+1} + c_1 + c_2 + \cdots + c_n,$$

and the corresponding *average* annual cost over these years is

$$a_n = \frac{1}{n}(v_1 - v_{n+1} + c_1 + c_2 + \cdots + c_n) \quad (n = 1, 2, \ldots). \tag{1.3}$$

The mathematical model is encapsulated in equations (1.1)–(1.3), together with the list of definitions of the variables which appear in these equations.

Before moving on to solve the equations, it is timely to list some of the assumptions which have been made, explicitly or implicitly, in the course of constructing the model.

- The value of the car at a given age is the same for both purchase and resale (there is no 'dealer margin' involved).
- The value of the car diminishes by a fixed proportion each year.
- The annual servicing and repair costs increase linearly with age.
- The car is sold after a whole number of years.
- Both car value and annual running costs depend *only* on the age of the car (and not, for example, on its mileage).
- Any effects of inflation are to be ignored, as are the annual costs of fuel, insurance and tax.

In terms of the modelling cycle, it is time now to 'do the mathematics', in order to solve equations (1.1)–(1.3). Before embarking on this task, there are some mathematical preliminaries to consider.

1.2 Mathematical interlude: sums

Towards the end of the last subsection, the expression

$$c_1 + c_2 + \cdots + c_n$$

appeared, to denote the sum of the first n terms of a sequence c_i $(i = 1, 2, \ldots)$. In general, a sum of consecutive terms of a sequence of numbers is called a **series**. Since series occur frequently in mathematics, and it can become unwieldy to use the '$+ \cdots +$' device when several sums are involved, an alternative and more compact notation exists to describe them. This is referred to as 'sum notation' or as 'sigma notation', since it makes use of the Greek upper-case letter Σ (sigma). Using this notation, the sum above can be expressed as

$$c_1 + c_2 + \cdots + c_n = \sum_{i=1}^{n} c_i,$$

The right-hand side here is read as 'the sum from i equals 1 to n of c_i'.

in which 1 and n are called the *limits* of the sum. (Some books print the sum as $\sum_{i=1}^{n} c_i$ when it appears in a line of text.)

Sometimes it is convenient to start a sum from a value of i other than 1, as you will see in the following example.

Example 1.1 Using sigma notation

Express each of the following sums using sigma notation:

(a) the sum of the first n positive integers;

(b) the sum of the cubes of the integers between 3 and 9, inclusive;

Chapter A1, Subsection 4.2

(c) the result which you saw in Chapter A1 for the sum of a finite geometric series, that is,

$$a + ar + ar^2 + \cdots + ar^n = a\left(\frac{1 - r^{n+1}}{1 - r}\right) \quad (r \neq 1).$$

Solution

(a) The sum is $1 + 2 + \cdots + n = \sum_{i=1}^{n} i$.

(b) The sum is $3^3 + 4^3 + \cdots + 9^3 = \sum_{i=3}^{9} i^3$.

Note that $ar^1 = ar$ and that $ar^0 = a \times 1 = a$. Also, the convention that $0^0 = 1$ is adopted so that the case $r = 0$ is covered.

(c) Using sigma notation on the left-hand side, we have

$$\sum_{i=0}^{n} ar^i = a\left(\frac{1 - r^{n+1}}{1 - r}\right) \quad (r \neq 1).$$

Comment

In these sums i could be replaced by j, for example. The sum in part (a) would then be $\sum_{j=1}^{n} j$. Also, each of these sums could start at a different value if the term being summed is adjusted accordingly. For example, the sum in part (b) could be written as $\sum_{i=0}^{6}(i+3)^3$.

Here are some sums for you to manipulate.

Activity 1.2 Using sigma notation

(a) Use sigma notation to express the sum of the squares of the integers between 5 and 13, inclusive.

(b) By first writing out the two sums involved, express

$$\sum 5^i \quad \sum 5^i$$

as a single sum (with one sigma).

(c) Show that $\sum_{i=1}^{4}(600 + 250i) = 600 \times 4 + 250\sum_{i=1}^{4} i$.

Solutions are given on page 55.

The outcome of Activity 1.2(c) is a special instance of a result that is often useful in simplifying the expression of sums. If x_i $(i = 1, 2, \ldots)$ is a sequence and a, b are constants, then we have

$$\sum_{i=1}^{n}(a + bx_i) = an + b\sum_{i=1}^{n} x_i. \tag{1.4}$$

More generally, if the sum starts at m (where $m \leq n$), then

$$\sum_{i=m}^{n}(a + bx_i) = a(n - m + 1) + b\sum_{i=m}^{n} x_i. \qquad (1.5)$$

In Example 1.1(c) you were reminded of the formula for the sum of a finite geometric series. There is also a formula for the sum of the finite arithmetic series $1 + 2 + \cdots + n$, that is, for the sum of the first n positive integers. This formula is derived by writing down the sum in the normal order and then with this order reversed.

$$\sum_{i=1}^{n} i = 1 + 2 + 3 + \cdots + (n-1) + n,$$

$$\sum_{i=1}^{n} i = n + (n-1) + (n-2) + \cdots + 2 + 1.$$

On adding these two equations, with the addition of corresponding terms on the right-hand side, we obtain

$$2\sum_{i=1}^{n} i = \underbrace{(n+1) + (n+1) + (n+1) + \cdots + (n+1) + (n+1)}_{n \text{ such terms in total}}$$
$$= n(n+1),$$

so the required sum is

$$\sum_{i=1}^{n} i = \tfrac{1}{2}n(n+1). \qquad (1.6)$$

This result will be useful in the next subsection. Note the geometrical interpretation of it, which is illustrated (for the case $n = 5$) in Figure 1.2. The total shaded area is equal to the sum of the shaded column areas, which is $1 + 2 + 3 + 4 + 5 = \sum_{i=1}^{5} i$, and also to half of the area of the whole rectangle, which is $\frac{1}{2} \times 5 \times 6$.

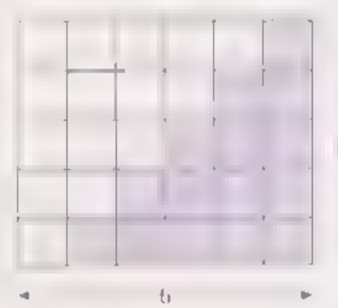


Figure 1.2 Shaded area equals half of area of rectangle

Activity 1.3 Summing positive integers

Find the sum of the first 100 positive integers.

A solution is given on page 55.

Intellectual precocity

It is said that the great German mathematician Carl Friedrich Gauss (1777–1855), at the age of ten, was asked along with the rest of his school class to carry out the sum which you did in Activity 1.3, having been shown no way of doing it beyond the addition of successive numbers. Their teacher intended it to be a lengthy task, but Gauss came up with the correct answer almost immediately, by applying the approach used above to derive equation (1.6).

'Gauss had not been shown the trick for doing such problems rapidly. It is very ordinary once it is known, but for a boy of ten to find it instantaneously by himself is not so ordinary. This opened the door through which Gauss passed on to immortality.'

(E. T. Bell, *Men of Mathematics*, Volume 1 (Pelican, 1953))

1.3 Back to the model

The mathematical model developed in Subsection 1.1 was described by equations (1.1)–(1.3). On using sigma notation to rewrite the sum in equation (1.3), we obtain the equations

$$v_0 = 16\,000, \quad v_{i+1} = 0.85 v_i \quad (i = 0, 1, \ldots, n), \tag{1.1}$$

$$c_0 = 600, \quad c_{i+1} = c_i + 250 \quad (i = 0, 1, \ldots, n-1), \tag{1.2}$$

$$a_n = \frac{1}{n}\left(v_1 - v_{n+1} + \sum_{i=1}^{n} c_i\right) \quad (n = 1, 2, \ldots). \tag{1.7}$$

Do the mathematics

Equations (1.1) define a geometric sequence, and equations (1.2) define an arithmetic sequence. Once a closed form has been found for each of these, the resulting expressions for v_n and c_n can be used on the right-hand side of equation (1.7). With a_n then expressed in terms of n, we can seek to minimise the value of a_n with respect to changes in n.

The closed forms for an arithmetic sequence and for a geometric sequence were given in Chapter A1, Subsections 2.2 and 3.2, respectively. It is common practice to refer to such formulas as 'closed-form solutions' of the underlying recurrence system.

Activity 1.4 Finding the closed-form solutions

(a) Find the closed-form solution of the recurrence system (1.1).

(b) Find the closed-form solution of the recurrence system (1.2).

(c) Use equations (1.4) and (1.6) to simplify as far as possible the expression for $\sum_{i=1}^{n} c_i$ that follows from your answer to part (b).

Solutions are given on page 55.

Comment

Any finite arithmetic series can be summed in a manner similar to that seen in the solution to part (c). Alternatively, such a sum can be found directly by application of the approach used to derive equation (1.6).

From the result of Activity 1.4(a), we have

$$\begin{aligned} v_1 - v_{n+1} &= 16\,000(0.85) - 16\,000(0.85)^{n+1} \\ &= 16\,000 \times 0.85(1 - (0.85)^n) \\ &= 13\,600(1 - (0.85)^n). \end{aligned}$$

From the result of Activity 1.4(c), we have

$$\sum_{i=1}^{n} c_i = 25n(5n + 29).$$

On substituting these formulas into equation (1.7), we obtain

$$\begin{aligned} a_n &= \frac{1}{n}(13\,600(1 - (0.85)^n) + 25n(5n + 29)) \\ &= \frac{13\,600}{n}(1 - (0.85)^n) + 25(5n + 29) \quad (n = 1, 2, \ldots). \end{aligned}$$

With a_n now expressed in terms of n, we seek the smallest value of a_n. This can be found by tabulating values of the sequence a_n, as shown below. (The values are given correct to two decimal places.)

n	1	2	3	4	5	6	7	8	9	10
a_n	2890	2862	2849.3	2850.18	2863.12	2886.79	2920.02	2961.77	3011.11	3067.25

It appears that the smallest value of a_n is 2849.3, which occurs when $n = 3$.

Interpret the results

The prediction of the model is that, having acquired a one-year-old vehicle of a certain type, the lowest average annual cost of owning and running it thereafter is obtained by selling it after a further three years, for which the average annual cost will be £2849.30 (not including fuel, insurance and tax).

In order to interpret the results obtained from a mathematical model, it is often helpful to draw a graph. Figure 1.3 shows graphically how a_n varies with changes in n.

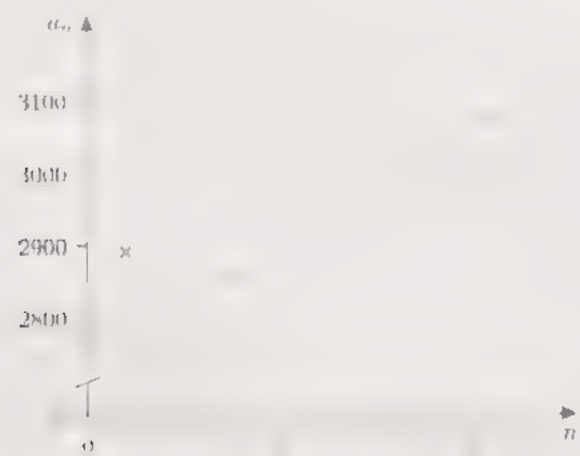


Figure 1.3 The sequence a_n

This graph indicates once more that the minimum value of a_n occurs for $n = 3$, but it also shows that this minimum is barely below the value for $n = 4$. This observation is substantiated by another look at the table. The average annual cost for resale of the car after four years is less than £1 more than the minimum cost identified, and this difference is not enough to decide whether to sell the car after three years or after four.

More generally, the variation of a_n around $n = 3$ is not huge. Perhaps you would not object to selling the car after two years or after five, for the average costs indicated. On the other hand, the increased costs involved in selling after one year or beyond six years give pause for thought.

The model tells you that, on the basis of the various assumptions made here, resale after anything between three and four years is the most economical approach available, at an estimated average cost of about £2850 per annum plus fuel, insurance and tax.

Evaluate the outcomes

On the face of it, the outcome of this modelling activity looks satisfactory. We set out to find a 'minimum cost of motoring', in specified circumstances, and have indeed found a feasible strategy for the car owner which achieves such a minimum. However, this has been done within the context of an idealised and simplified 'world view', geared particularly to bringing mathematics to bear on the situation. The hope is that this approach has revealed some essential truth about the situation being studied, but this cannot yet be taken for granted. Before accepting the conclusions from our model, it would be wise to check that the course of action proposed (resale after three to four years) makes sense in reality.

We do not have sufficient information to make that check with reality here, but it may well turn out (it is frequently the case) that the initial model is judged as only partly successful once the comparison with reality has taken place. If so, it may be necessary to try to improve the model, by taking another trip around the five stages of the modelling cycle. An early step would be to re-examine the assumptions which were made in constructing the first model, in order to see whether any need amendment in the light of experience gained.

For the model developed here, for example, the assumption of linearly increasing annual costs for servicing and repairs may turn out to be too simple. The data underlying this assumption are few, and more data on annual running costs might suggest a suitable refinement.

Generalising the model

The problem posed in the 'Purpose of model' box on page 9 has been solved, to the extent possible with the information available. Note, however, that once a model has been constructed with such a specific purpose in mind, it may come in useful for other related purposes. In this case, we set up a model for one particular type of car, but the *structure* of the model would be unchanged if the same approach were applied to a different type of car.

The remainder of this section will not be assessed.

Thus, in place of equations (1.1), we have a formula for car value given by

$$v_{i+1} = rv_i \quad (i = 0, 1, \ldots, n), \tag{1.8}$$

In fact, it would have been better modelling practice to proceed with these more general formulas in the first instance, instead of introducing illustrative data values as was done in Subsection 1.1.

where r is a constant, and in place of equations (1.2), we have a formula for servicing and repair costs given by

$$c_{i+1} = c_i + d \quad (i = 0, 1, \ldots, n-1), \tag{1.9}$$

where d is a constant. Also, equation (1.7) continues to apply:

$$a_n = \frac{1}{n}\left(v_1 - v_{n+1} + \sum_{i=1}^{n} c_i\right) \quad (n = 1, 2, \ldots). \tag{1.7}$$

Activity 1.5 Doing the mathematics for the general case

This activity requires a generalisation of what you did in Activity 1.4 and the main text following that activity.

(a) Find the closed-form solution of the recurrence relation (1.8).

(b) Find the closed-form solution of the recurrence relation (1.9).

(c) Use equations (1.4) and (1.6) to simplify as far as possible the expression for $\sum_{i=1}^{n} c_i$ which follows from your answer to part (b).

(d) Starting from equation (1.7) and your answers to parts (a) and (c), express a_n as simply as possible in terms of the variable n and the parameters v_0, r, c_0 and d.

Solutions are given on page 55.

Comment

In the final formula obtained in part (d), parameter values can be substituted for v_0, r, c_0 and d to correspond to any given make and model of car. The resulting specific version of the formula for a_n can then be used to predict when such a car should be resold, after purchase at one year old, to most economical effect.

Further generalisations of the model are possible, for example, to take into account purchases at age other than one year. However, we shall not pursue such extensions here.

Summary of Section 1

This section has introduced:

- the use of sigma notation to describe concisely sums of terms from a sequence;
- the word 'series' to describe a sum of consecutive terms of a sequence;
- the formula $\frac{1}{2}n(n+1)$ for the sum of the first n positive integers;
- the application of general modelling ideas to a specific problem involving the economics of car ownership.

You will not be assessed directly on the modelling content of this section, though the principles are important if you want to continue to further studies in applied mathematics. The framework for the modelling cycle, within the five key stages shown in Figure 1.1, was described concisely in Chapter A1, Section 7. To some extent, at least, all of the aspects included there were addressed in the course of the modelling done in this section.

Exercises for Section 1

Exercise 1.1

(a) The sum

[illegible]

can be expressed as $\sum_{i=1}^{10}(\frac{1}{2})^i$, and also in the form

[illegible]

What are the values of m and n?

(b) What is the value of this sum?

Exercise 1.2

(a) Find the sum of the integers from 51 to 100, inclusive.

(b) Hence find the sum of the numbers $517, 527, 537, \ldots, 1007$.

(*Hint*: $517 = 7 + 10i$, where $i = 51$.)

2 Populations: exponential model

2.1 Purpose of general population models

What will happen to the North Atlantic population of tuna if fishing continues at its present levels? What strategies are effective in controlling the spread of rabies in foxes? Will the blue whale escape extinction? Can the world support 10 billion humans?

A mathematical model which aims to answer any of these questions about animal populations would need to be based on detailed information about the species concerned. However, to create a model to address any *particular* question, one would draw on *general* models of the way an animal population can change with time. The purpose of these models might be stated thus.

Purpose of model

Describe how an animal population may change with time, in a way that is applicable over a long period of time and for a variety of populations.

We shall pay particular attention to the form of population variation that is predicted for the long term. Will the population increase, perhaps more and more rapidly? Or will it decrease, perhaps to extinction? Or will the population stabilise, and if so at what level?

In Section 1 you saw a single model developed to answer a quite specific question. In this and the next section, you will meet two successive models set up in order to address the rather general purpose just stated. These are 'standard models' used in the study of animal populations, and you are not asked to dwell on the modelling processes which lead to their construction, though you should be able to recognise that each of the various modelling stages shown in Figure 1.1 is involved. The second model, in Section 3, is a development of the first, in this section, and arises from some perceived deficiencies of the first model.

2.2 The exponential model

For many animal populations, there is a pattern of variation within each year. Population sizes are highest in the early autumn, after births during spring and summer, and lowest at the end of winter, during which most deaths occur. In studying long-term variations in population levels, the main interest is in changes from one year to the next, rather than within a year. We therefore look here at discrete (rather than continuous) models of population change.

For details of the distinction between discrete and continuous models, see the Summary of Block A at the end of Chapter A3

For example, Figure 2.1 shows the size of a population of pheasants on 1 April and 1 October each year for 1937 to 1942. You can see that numbers dropped during each winter, even though they increased from one year to the next. We seek a model that predicts this year-on-year increase.

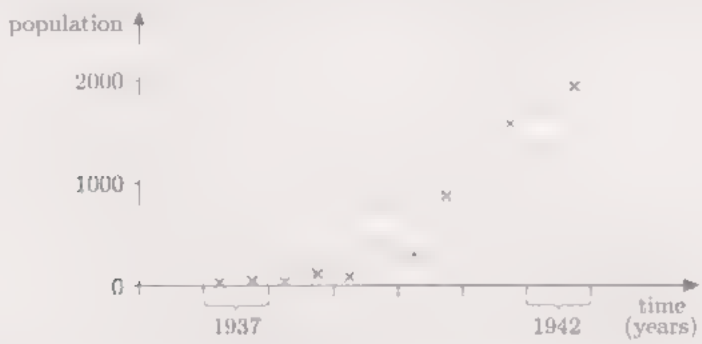


Figure 2.1 Population of ring-necked pheasants on Protection Island, USA, in spring and autumn (1937 1942)

In setting up population models, a useful approach is to identify the numbers of a species which leave or join the population each year. The 'joiners' include those who are born during the year, while the 'leavers' include those who die. Changes in population size may also be brought about by geographical movements between separate populations of animals of the same species (immigration and emigration). In this study, we confine attention to populations where migration is not an important factor. The models here assume that there is *no migration*, so we need to focus on only the effects of births and deaths.

For human populations, a birth rate is usually given as a proportion of the current population size (perhaps as a percentage, or as 'births per thousand'), and death rates are specified in a similar way. It seems natural that for any animal population, the numbers of births and deaths will both increase as the population size increases. As a first model of population variation, it seems sensible to assume that each of the number of births and the number of deaths is a fixed proportion of the current population size. This model is illustrated in the following example.

Example 2 1 Modelling pheasants

Let P_n denote the population size of ring-necked pheasants on Protection Island on 1 April, n years after 1937; thus P_0 represents the population size on 1 April 1937, which is 8. Assume that the number of births in each subsequent year (from 1 April to 31 March) is $2.6P_n$ (that is, 260% of P_n), and the number of deaths in each subsequent year is $0.4P_n$ (40% of P_n).

(a) Find a recurrence system that P_n must satisfy.

(b) State a closed form for P_n.

(c) What population sizes does this model predict for 1 April in each of the years 1938, 1939, ..., 1942?

(d) What form of population variation does this model predict in the long term?

Solution

(a) During the $(n+1)$th year, the 'joiners' will be the pheasants born during the year, of which there are $2.6P_n$, and the 'leavers' will be the pheasants that die, of which there are $0.4P_n$. The difference between births and deaths gives the increase in the population size during the year (since we are assuming that there is no migration), so

$$\begin{aligned} P_{n+1} - P_n &= 2.6P_n - 0.4P_n \\ &= 2.2P_n. \end{aligned}$$

Thus the sequence P_n is given by

$$P_0 = 8, \quad P_{n+1} = 3.2P_n \quad (n = 0, 1, 2, \ldots).$$

In the population contexts considered from now on, the subscript range will always be $n = 0, 1, 2, \ldots$, as here, and it will usually be omitted in this text.

(b) This recurrence system describes a geometric sequence (as studied in Chapter A1), with closed form

$$P_n = 8(3.2)^n \quad (n = 0, 1, 2, \ldots).$$

(c) The predicted population sizes on 1 April are shown below. (Each estimate is calculated using unrounded intermediate values, then rounded to the nearest whole number.)

Year	1937	1938	1939	1940	1941	1942
Population size	8	26	82	262	839	2684

(d) This model predicts that the population size will increase, and will go on increasing more and more rapidly (see Figure 2.2).

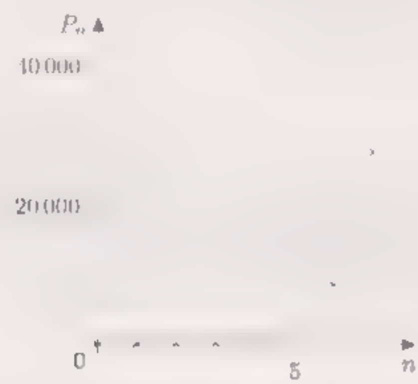


Figure 2.2 Graph of $P_n = 8(3.2)^n$ $(n = 0, 1, \ldots, 7)$

Comment

This model predicts that the population size will increase very rapidly, and this pheasant population *did* undergo rapid increase from 1937 to 1942 (as shown in Figure 2.1). However, the model predicts that this increase will continue. For example, it predicts a population size on 1 April 1997 of $8(3.2)^{60}$, which is about 1.6×10^{31}. Estimating the area of ground taken up by a pheasant as $0.1\,\mathrm{m}^2$, and using the fact that the island has area $1.6 \times 10^6\,\mathrm{m}^2$, this gives $1.6 \times 10^{30}\,\mathrm{m}^2$ of pheasants in an area of $1.6 \times 10^6\,\mathrm{m}^2$. This implies that the island will be covered about 10^{24} deep in pheasants, which seems unlikely!

If the purpose of the model in this example is to predict the population size in the long term, then it is certainly not reasonable. The population cannot continue to grow indefinitely in the way predicted by the model.

We can generalise the model in Example 2.1 to other populations, by keeping the assumptions that the birth and death rates are both constant, but treating these values as parameters. We shall now investigate whether such a generalised model predicts this same form of population growth in the long term.

Suppose that P_0 is the size of a particular population at the start of a period of time for which the population is to be modelled, and that P_n denotes this population size at n years after the starting time. Suppose that the proportionate birth rate b and death rate c are both (non-negative) constants, so that, during the year starting at time n, there are bP_n births and cP_n deaths. Suppose, as before, that there is no migration.

For the year starting at time n, the change in population size during the year is $P_{n+1} - P_n$, and this must equal 'joiners' minus 'leavers'. So we have

$$P_{n+1} - P_n = bP_n - cP_n,$$

which is equivalent to

$$P_{n+1} = (1 + b - c)P_n.$$

This again describes a geometric sequence, for which the closed form is

$$P_n = (1 + b - c)^n P_0.$$

The long-term behaviour of P_n will depend on the value of $1 + b - c$.

Since the death rate c is never greater than 1,

$$1 + b - c \geq b \geq 0.$$

If $1 + b - c = 0$, then $P_n = 0$ for $n = 1, 2, 3, \ldots$; otherwise, $P_n > 0$, unless $P_0 = 0$.

Activity 2.1 Predicted behaviour of exponential model

Describe the long-term behaviour of $P_n = (1 + b - c)^n P_0$ when $1 + b - c > 0$.

A solution is given on page 56.

The three cases identified in Activity 2.1 are illustrated in Figure 2.3.

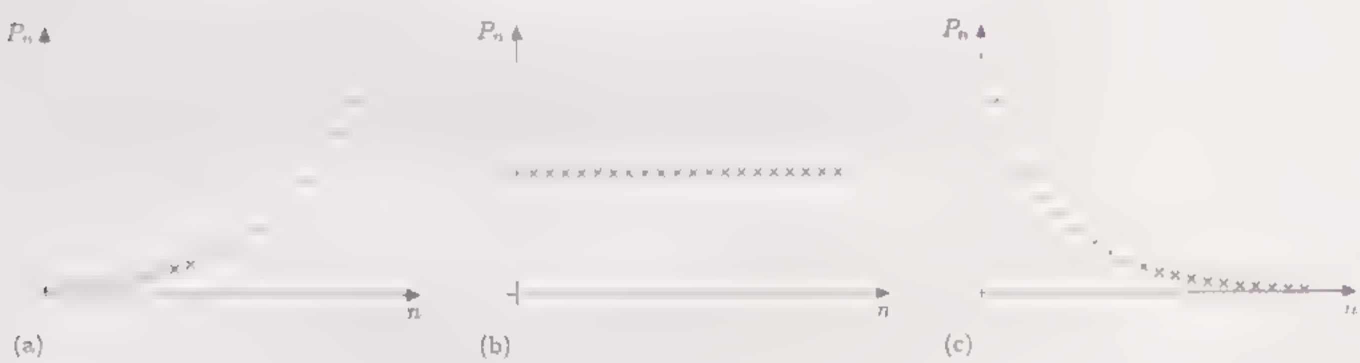


Figure 2.3 Plots of $P_n = (1 + b - c)^n P_0$ for (a) $b > c$, (b) $b = c$, (c) $b < c$

The parameters in this model are the starting population size, P_0, the proportionate birth rate, b, and the proportionate death rate, c. Since the last two of these occur in the combination $b - c$, it is often convenient to replace this combination by a single symbol, for which we shall choose r.

This growth rate r gives the net effect of births minus deaths over each year, per head of the population. Although r is a *growth* rate, it can be negative.

Thus $r = b - c$ is the annual **proportionate growth rate** of the population. In terms of r, the recurrence relation for P_n is

$$P_{n+1} = (1 + r)P_n,$$

and the closed-form solution is

$$P_n = (1 + r)^n P_0.$$

The graphs in Figure 2.3 are those of $P_n = (1 + r)^n P_0$ in each of these cases.

The three cases $b > c$, $b = c$ and $b < c$ correspond respectively to $r > 0$, $r = 0$ and $r < 0$.

The model just derived could be called a *geometric model* for population variation, since it predicts that the population size rises or falls according to a geometric sequence. However, it is more usual to describe it as the *exponential model*, since the expression $(1 + r)^n P_0$ involves raising the base number $1 + r$ to the exponent n.

Exponential model

The exponential model for population variation is based on the assumption of a constant proportionate growth rate, r. The model is described by either the recurrence relation

$$P_{n+1} = (1 + r)P_n \quad (n = 0, 1, 2, \ldots), \tag{2.1}$$

or its closed-form solution

$$P_n = (1 + r)^n P_0 \quad (n = 0, 1, 2, \ldots), \tag{2.2}$$

where P_n is the population size at n years after some chosen starting time.

Given data on population size over at least several years, it is possible to select the parameters P_0 and r in such a way that the corresponding exponential model (given by equation (2.2)) 'fits the data' as closely as possible. We shall not be going into the full details of how this is done, but the following example and activity show what is involved in idealised cases.

Example 2.2 Modelling gannets

A colony of northern gannets at Cape St Mary, Newfoundland, Canada, numbered 26 in the year 1889 and 160 in 1899.

(a) Assuming that this population can be modelled exactly over the intervening period by an exponential model, find the value of the annual proportionate growth rate, r, to two significant figures.

(b) Write down the corresponding formula for the population size P_n of the colony at n years after the census date in 1889.

(c) Assuming that this model continues to hold after 1899, what population size is predicted for the colony in the year 1914

Solution

(a) Take $n = 0$ at the census date in 1889. Then $n = 10$ in 1899, so $P_0 = 26$ and $P_{10} = 160$. According to the exponential model (equation (2.2)), we have $P_n = (1 + r)^n P_0$, so, in particular,

$$P_{10} = (1+r)^{10} P_0; \quad \text{that is,} \quad 160 = (1+r)^{10} \times 26.$$

It remains to solve this equation for the proportionate growth rate, r. On dividing through by 26 and then taking the 10th root, we obtain

$$1 + r = \sqrt[10]{\frac{160}{26}}, \quad \text{so} \quad r = \left(\frac{160}{26}\right)^{0.1} - 1 = 0.20 \quad \text{(to 2 s.f.).}$$

Hence the proportionate growth rate is about 20% per year.

(b) The exponential model for the colony is therefore $P_n = 26(1.20)^n$.

(c) The year 1914 corresponds to $n = 25$, for which the population size of the colony is predicted to be $P_{25} = 26(1.20)^{25} \simeq 2480.30$. This must be rounded to a whole number, to provide a meaningful prediction for a population size. The prediction for the year 1914 is therefore 2480.

(Note that even rounding to the nearest whole number may be misleading. This prediction is based on a value of r which was rounded to two significant figures, so there is reason to question its accuracy. If we use the *exact* value of r to calculate P_{25}, then the prediction becomes 2443 to the nearest integer.)

As a check, the value of $26(1.20)^{10}$ should be close to 160, the given population size for 1899. In fact,

$$26(1.20)^{10} \simeq 160.99.$$

We shall not pursue the effects of such rounding. In further discussion of this model, we take $r = 0.20$.

Activity 2.2 Modelling the US population (1790–1890)

The (human) US population was about 4 million in 1790, when the first national census was taken, and 63 million in 1890.

(a) Assuming that this population can be modelled exactly over the intervening period by an exponential model, find to two significant figures the value of the annual proportionate growth rate, r. (Take the population to be measured in millions.)

(b) Write down the corresponding formula for the US population P_n (in millions) at n years after the census date in 1790.

(c) Assuming that this model continues to hold for years after 1890, what US population is predicted in the year 1950?

(d) In which year, according to the model, does the US population reach 500 million?

The approach needed to answer this part is similar to that for problems in Chapter A3, Section 4: for example, Activity 4.6(b) and the main text before that activity.

Solutions are given on page 56.

2.3 Evaluating the exponential model

To complete the modelling cycle for the general exponential model developed in the previous subsection, we need to evaluate it. Is this model satisfactory for the stated purpose, which was to describe how numbers in a population may change with time, in a way that is applicable over a long period of time and for a variety of populations? We consider in turn the three cases identified in Activity 2.1 and illustrated in Figure 2.3.

If $b < c$ (that is, $r < 0$), then the prediction is for a decreasing population size. If a population has fewer births than deaths, then it can be expected to decrease, and examples of real populations varying in this sort of way can be given (see Figure 2.4). In this case, the mathematical solution $P_n = (1+r)^n P_0$ will eventually predict populations of size less than 1. Such a prediction is not to be taken literally, of course. Values of P_n less than 1 would suggest that the population has reached extinction. In practice, a model implying such population decrease would be interpreted as predicting extinction when the population size drops below some minimum level from which there is no chance of recovery. There is nothing obviously unrealistic about the general form of population change predicted in this case.

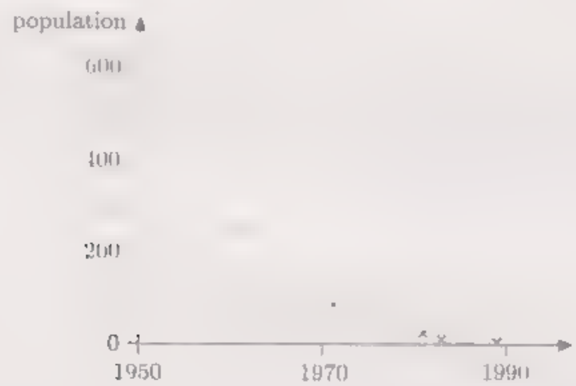


Figure 2.4 Population (in breeding pairs) of red-backed shrikes in Great Britain (1952–1989)

If $b = c$ (that is, $r = 0$), then the prediction is for a constant population. This may seem to be a peculiar 'special case', but in fact it is the closest to reality in many instances. A fairly steady population is more typical than rapid increase or decrease. An example is given in Figure 2.5(a).

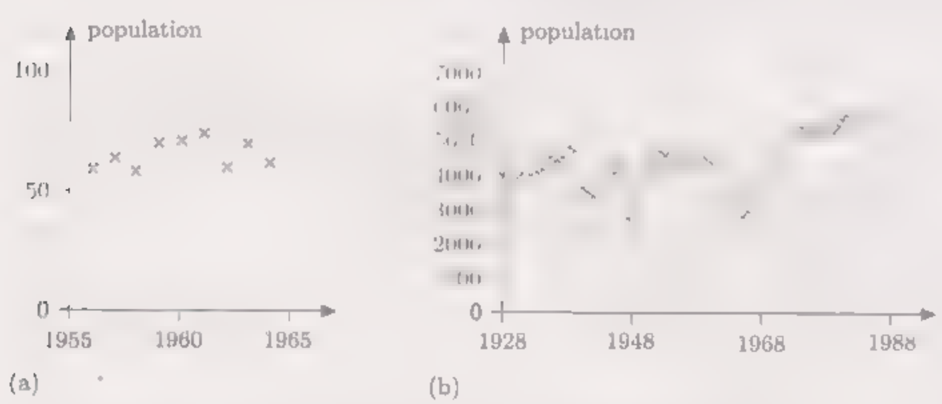


Figure 2.5 Examples of fairly steady populations (in breeding pairs):
(a) pied flycatchers at Lemsjöholm, Germany (1956–1964),
(b) grey herons in England and Wales (1928–1986)

The graph in Figure 2.5(b) shows fairly steady population size over some periods, interspersed with sudden drops and subsequent steady recoveries.

Here we must recall that the exponential model relies on the assumption of *constant* proportionate birth and death rates. This assumption will fail to hold where there are marked variations in weather or other conditions which could affect the population. It is noticeable that the largest sudden drop for the grey heron population of Figure 2 5(b) coincides with the particularly severe UK winter of 1963

If $b > c$ (that is, $r > 0$), then the prediction is for a population that increases. This increase continues indefinitely, and becomes more and more rapid. The prediction in this case is not reasonable as regards long-term behaviour. Any population which is subject to unlimited increase will eventually reach an unsustainable size. What constitutes an 'unsustainable' level, and how long the population may take to reach that, depends on the particular population being considered.

Many human populations have shown exponential increase over quite long periods of time, as for example in the case of the US population considered in Activity 2.2. Figure 2.6(a) shows the US population for the period 1790 1890, together with the exponential model whose growth rate was obtained in Activity 2.2(a). The fit between the model and the data appears good here. However, Figure 2.6(b) looks at the US population on a different time scale, and includes further census data up to 1990. It is evident that the previous exponential model is far from accurate in its predictions after 1900. For example, the prediction from Activity 2.2(c) of a population size 332 million in the year 1950 is more than twice as large as the actual figure for that year, 151 million.

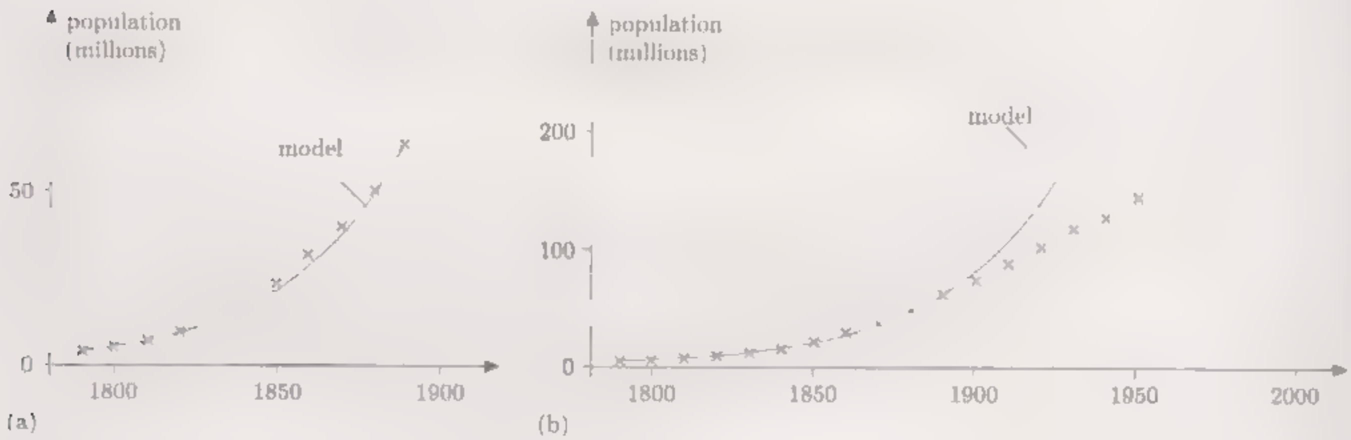


Figure 2.6 US population: (a) 1790–1890, (b) 1790–1990

The same inaccuracy applies for the gannet colony considered in Example 2.2. The fit to data of the exponential model looks quite good at relatively low population levels, but not thereafter (see Figure 2.7(a), overleaf). The rate of increase slows down in later years (at higher population levels). Such a slowing down of the rate of increase is also apparent for the grey herons after 1963 (Figure 2.5(b)) and for the populations shown in Figure 2.7(b) and (c). These three diagrams also suggest that some steady population size may eventually be reached.

The explanation of these last examples is that rapid growth of a population may occur when an animal species arrives in a new and favourable habitat, but that such growth will not be sustained indefinitely.

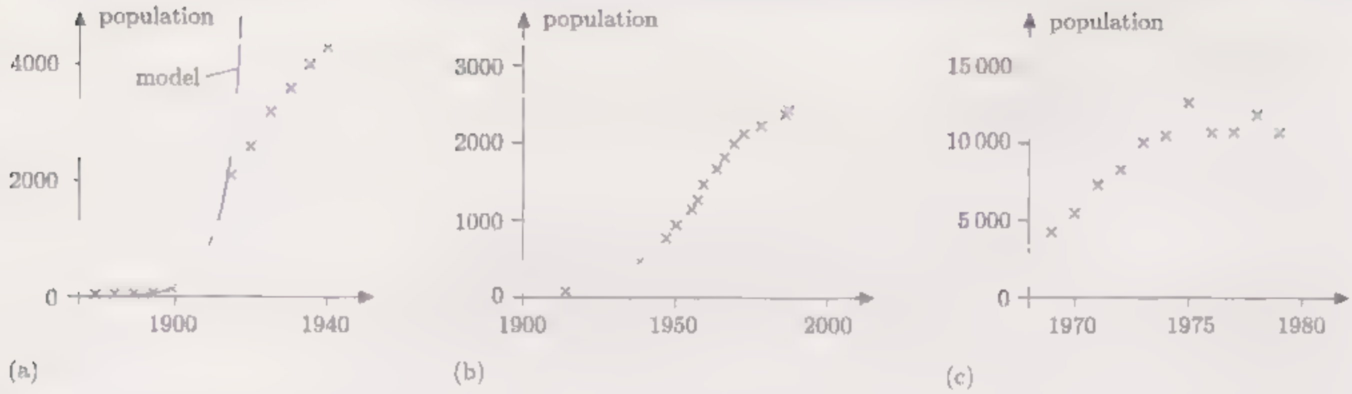


Figure 2.7 Examples of increasing populations.
(a) population (in breeding pairs) of northern gannets at Cape St Mary, Newfoundland, Canada [illegible]
(b) estimates of the sea otter population off central California, USA (1914 1987),
(c) winter counts of elk in North Yellowstone National Park, USA (1968 1979)

Other types of population variation are also found with some species: examples are shown in Figure 2.8. (For illustrative purposes, it is assumed that the flour beetles of Figure 2.8(a) breed in clearly separated generations.) Clearly, the exponential model has little application in such cases.

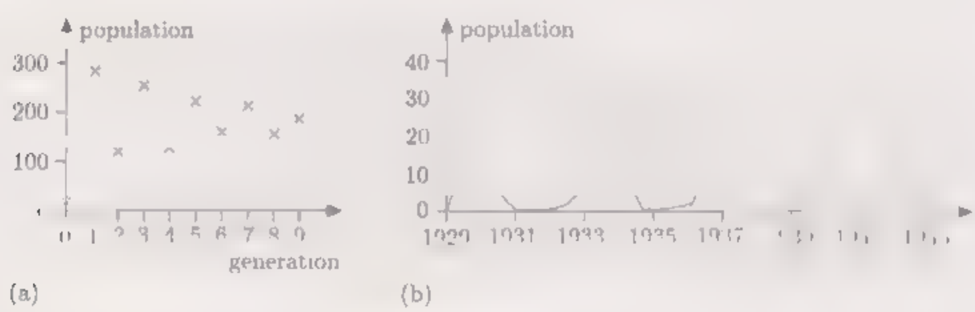


Figure 2.8 Some other examples of population variation:
(a) numbers in successive generations of a laboratory population of a flour beetle,
(b) population density of [illegible] near Churchill, Canada [illegible]

A population density is the number per unit area.

We conclude that the exponential model may be appropriate in certain instances (declining populations, steady size, increases from a low level), but that it has a serious flaw in predicting unlimited growth. It is this flaw which we attempt to rectify by revising the model in the next section.

Origins of the exponential model

The exponential (or geometric) model for population variation is also referred to as *Malthusian*, after Thomas Robert Malthus (1766–1834), who was for many years Professor of Political Economy and Modern History at the College of the East India Company in the UK. In 1798 he published *An Essay on the Principle of Population*, which expressed his belief that the natural effect of human reproduction, if unconstrained, was to produce geometric increases in population size. He also thought that food production could increase only at an arithmetic rate, which suggested a future in which starvation might be the major factor in limiting populations. To avoid this bleak prospect, he suggested that attention should no longer be paid to improving living conditions for the poor, or to seeking advances in medical science (so as not to lower death rates), while birth control and moral restraint were to be encouraged. (However, Malthus himself had 11 children!)

The views of Malthus had considerable influence on social policy at the time, and may well have been partly responsible for the undertaking of the first UK census in 1801

Summary of Section 2

This section has introduced:

- the exponential (or geometric) model for population variation, $P_{n+1} = (1+r)P_n$, which is based on the assumption of a constant proportionate growth rate r and has the closed-form solution $P_n = (1+r)^n P_0$, where P_n is the population size at n years after some chosen starting time;
- a method for estimating, from data about the population, a value for the parameter r in the exponential model;
- evidence of the invalidity of the exponential model for large population sizes, in that it predicts unlimited growth.

Exercises for Section 2

Exercise 2.1

According to UN estimates, the (human) population of the world was 1.65 billion in the year 1900 and 2.52 billion in 1950.

(a) Assuming that this population can be modelled exactly over the intervening years by an exponential model, find to two significant figures the annual proportionate growth rate for the period 1900–1950.

(b) Assuming that this model continues to hold after 1950, what world population is predicted for the year 2000?

The world is one habitat for which we can be sure that there is no external migration!

Note that, in UN statistics,

$$1 \text{ billion} = 1000 \text{ million}.$$

Exercise 2.2

In fact, growth of the world population has been considerably faster since 1950 than the model of Exercise 2.1 would suggest. Using the estimated world population of 6.06 billion in the year 2000, repeat the task of Exercise 2.1(a) for the period 1950–2000.

According to the UN, the world population reached 6 billion in October 1999.

3 Populations: logistic model

The general exponential model for population variation, developed in Subsection 2.2, is summed up in the recurrence relation

$$P_{n+1} = (1 + r)P_n \tag{2.1}$$

and the corresponding closed-form solution

$$P_n = (1 + r)^n P_0. \tag{2.2}$$

This model was seen to work well enough for relatively small populations, but to become invalid for large population sizes. We now seek to amend the model in the case of a *positive* proportionate growth rate r, in order to cope more realistically with the effects of continued population growth.

3.1 Setting up the logistic model

Equation (2.1) was based on the premise that both b (the proportionate birth rate) and c (the proportionate death rate) were constant, regardless of population size, so that the same was true of the proportionate growth rate, $r = b - c$. Observation of various species suggests strongly that, all other factors being equal, birth rates tend to decline or death rates to increase (or both) as population size rises. These trends may be thought of as the consequence of competition between individuals of the species within a habitat of limited resources. They both point to a decline in the proportionate growth rate as population size increases. We now seek to build this feature into a revised model for population variation.

Here 'in year n' refers to a particular fixed census date within the year concerned.

It may be timely at this point to recall exactly what 'proportionate growth rate' means. If a population has size P_n in year n, and size P_{n+1} in year $n + 1$, then the *growth rate* of that population over the year from n to $n + 1$ is $P_{n+1} - P_n$. The *proportionate growth rate* is the 'increase per head', which is the actual growth rate divided by the population size at the start of the year:

$$\frac{P_{n+1} - P_n}{P_n}.$$

The exponential model assumes that the proportionate growth rate is a constant, r, so that

As you can check, this equation is just a rearrangement of equation (2.1).

$$\frac{P_{n+1} - P_n}{P_n} = r.$$

We now seek to alter this assumption, and put

R is a real function.

$$\frac{P_{n+1} - P_n}{P_n} = R(P_n), \tag{3.1}$$

where $R(P)$ is a function of population size P. We noted two points above:

- that the exponential model, based on $R(P) = r$ (constant), works well when P is relatively small;
- that the proportionate growth rate, $R(P)$, is seen to decrease, in practical situations, as P increases.

The simplest modelling step which takes account of these observations is to

assume that the proportionate growth rate, $R(P)$, is a decreasing *linear* function of the form

$$R(P) = mP + r,$$

where $m < 0$ (to meet the second point) and $R(P) \simeq r$ when P is small (to incorporate the first one).

This function may be expressed as

$$R(P) = r\left(1 - \frac{P}{E}\right), \qquad (3.2)$$

where r and E are positive parameters. (Recall that we are seeking to amend the exponential model in the case $r > 0$.) The graph of this function is given in Figure 3.1, which shows that r and E are the intercepts of the line on the axes.

Figure 3.1 Graph of $R(P) = r(1 - P/E)$

On substituting the expression for $R(P)$ from equation (3.2) into equation (3.1), we obtain

$$\frac{P_{n+1} - P_n}{P_n} = r\left(1 - \frac{P_n}{E}\right)$$

from which we conclude that, according to our revised model, P_n satisfies the recurrence relation

$$P_{n+1} - P_n = rP_n\left(1 - \frac{P_n}{E}\right). \qquad (3.3)$$

This equation is called the *logistic recurrence relation*, and the corresponding mathematical model is known as the *logistic model* for population variation.

Logistic model

The logistic model for population variation is based on the assumption of a proportionate growth rate $R(P)$ of the form

$$R(P) = r\left(1 - \frac{P}{E}\right), \qquad (3.2)$$

where r and E are positive parameters.

The model is described by the recurrence relation

$$P_{n+1} - P_n = rP_n\left(1 - \frac{P_n}{E}\right) \quad (n = 0, 1, 2, \ldots), \qquad (3.3)$$

where P_n is the population size at n years after some chosen starting time.

We have now completed the second stage in the modelling cycle for the logistic model. It is useful at this point to review how far we have come, by looking in turn at each of the following questions.

- What do the variables in equation (3.3) represent?
- What assumptions are made in the model?
- How can we interpret the model's parameters in terms of the situation being modelled?

Variables

Sometimes P_n may be taken to denote the number of breeding pairs, rather than individuals, or to denote population density.

The time variable, n years, can be chosen to be zero at any convenient point in time. The population size at that time is P_0, and P_n then denotes the population size n years later. In fact, this can be generalised to other time intervals. We usually want to look at population sizes at yearly intervals, because of the pattern of population variation within each year that was mentioned at the start of Subsection 2.2, but there are occasions when a time interval other than one year is appropriate. This is particularly likely when considering laboratory experiments, which are not affected by nature's annual cycle. For the flour beetles in Figure 2.8(a), it would be natural to take P_n to be the population in the nth generation from the start of the experiment. The steps involved in creating the model do not depend on the choice of time interval, so this may be altered as appropriate to suit the circumstances of the population being studied.

Assumptions

As in the exponential model, we have assumed that migration is not a significant factor for the populations considered here. In moving from the exponential to the logistic model, we assumed that the proportionate growth rate is a decreasing linear function of population size. Since the growth rate is births minus deaths, the linear decline in proportionate growth rate could arise in several ways:

- proportionate birth rate decreases linearly with population size, while proportionate death rate remains constant;
- proportionate death rate increases linearly with population size, while proportionate birth rate remains constant;
- proportionate birth rate decreases linearly with population size, and proportionate death rate increases linearly with population size.

In each case, it is possible to derive the appropriate form of the logistic recurrence relation from suitable information on the birth and death rates of a population, as in the following example and activity.

Example 3.1 Modelling barnacle geese

In 1970 a winter refuge was created at Caerlaverock, Dumfries, Scotland, for the population of barnacle geese which spends its winters there. The size of this population grew steadily from about 3000 in 1970 to about 13 000 in the mid-1990s. Observations of the birth and death rates of the barnacle geese suggest that the following information describes the behaviour of the population during this period:

- the annual proportionate death rate is constant at 0.11;
- the annual proportionate birth rate decreases linearly with the population size, P, according to the formula $0.31 - 1.5 \times 10^{-5}P$.

The population size on 1 January 1970 was 3200.

(a) Find a recurrence system for P_n, the population size of barnacle geese n years after 1 January 1970.

(b) Show that the recurrence relation obtained is logistic, by identifying the values of the parameters r and E from equation (3.3) which apply in this case.

Solution

(a) The proportionate growth rate $R(P)$ at population size P is the proportionate birth rate minus the proportionate death rate; that is,

$$R(P) = (0.31 - 1.5 \times 10^{-5}P) - 0.11 = 0.20 - 1.5 \times 10^{-5}P.$$

Hence the population growth for the year which starts at time n is

$$R(P_n)P_n = (0.20 - 1.5 \times 10^{-5}P_n)P_n.$$

Since this is also the increase from P_n to P_{n+1}, we have the recurrence system

$$P_0 = 3200, \quad P_{n+1} - P_n = (0.20 - 1.5 \times 10^{-5}P_n)P_n.$$

(b) To show that this recurrence relation is logistic, the right-hand side must be written in the form $rP_n(1 - P_n/E)$:

$$\begin{aligned}(0.20 - 1.5 \times 10^{-5}P_n)P_n &= 0.20P_n\left(1 - \frac{1.5 \times 10^{-5}}{0.20}P_n\right)\\ &= 0.20P_n\left(1 - 7.5 \times 10^{-5}P_n\right).\end{aligned}$$

This is indeed of the required form, with $r = 0.20$ and $1/E = 7.5 \times 10^{-5}$, that is, $E = 13\,300$ (to 3 s.f.).

Activity 3.1 Revised model for the pheasants

A population of ring-necked pheasants was the subject of Figure 2.1 and Example 2.1 (page 19), where an exponential model was assumed. The following assumptions are now to be made in describing the behaviour of this population:

◇ the annual proportionate death rate is constant at 0.4;

◇ the annual proportionate birth rate decreases linearly with the population size, P, according to the formula $2.65 - 0.0015P$.

The assumption on the death rate is the same as in Example 2.1, but the assumption on the birth rate has been altered.

The population size on 1 April 1937 was 8.

(a) Find a recurrence system for P_n, the population size of ring-necked pheasants n years after 1 April 1937.

(b) Show that the recurrence relation obtained is logistic, by identifying the values of the parameters r and E from equation (3.3) which apply in this case.

Solutions are given on page 56.

Parameters

The parameter E may be interpreted from two different viewpoints. It arose in Figure 3.1, as the intercept of the proportionate growth rate graph (assumed linear) on the P-axis. Hence it is the *population size at which the proportionate growth rate is zero.*

On the other hand, it can be seen directly from the logistic recurrence relation (3.3) that if P_n attains the size E when $n = m$, then

$$P_{m+1} - E = rE\left(1 - \frac{E}{E}\right) = 0, \quad \text{so} \quad P_{m+1} = E.$$

Hence $P_{m+2} = E$, $P_{m+3} = E$.

In other words, if P_n reaches size E, then it stays there.

These two views are, of course, consistent. Zero growth rate implies a constant size of population, and vice versa. The parameter E is called the **equilibrium population level** (or, in some contexts, the **carrying capacity**).

More generally, the constant sequence

$$P_n = E \quad (n = 0, 1, 2, \ldots)$$

is a solution of the logistic recurrence relation. In fact, this sequence is the only non-zero constant sequence which satisfies the logistic recurrence relation. For if $P_n = c$ $(n = 0, 1, 2, \ldots)$, where c is a constant, then from equation (3.3) we have

$$c - c = rc\left(1 - \frac{c}{E}\right), \quad \text{so} \quad c = 0 \text{ or } c = E.$$

The constant sequence

$$P_n = 0 \quad (n = 0, 1, 2, \ldots)$$

represents a total absence of the population.

We do not yet know whether other sequences generated by the logistic recurrence relation will settle down near some particular positive value in the long term, but if they do so, then that value must be E. The population is then 'in equilibrium'.

Thus, according to the logistic model, the equilibrium population level for the barnacle goose population of Example 3.1 is 13 300, while the equilibrium population level for the ring-necked pheasants in Activity 3.1 is 1500.

The meaning of the parameter r is apparent from the way in which the logistic model was constructed: it is the *proportionate growth rate of the population at small population sizes.* We can use the interpretation of E given above to be more precise about what 'small' means here. If P_n is small *compared with the equilibrium population level*, E, then P_n/E is small compared with 1, so that the logistic recurrence relation (3.3) is approximated closely by

$$P_{n+1} = (1 + r)P_n,$$

which is the exponential model once again. We have demonstrated that this more primitive model is a limiting case of the logistic model. At low population levels, population growth can be modelled satisfactorily by the exponential model formula, $P_n = (1 + r)^n P_0$, but this continues only while the population size remains well below its equilibrium level, E.

3.2 Finding values for the parameters

You have seen (in Example 3.1 and Activity 3.1) how to deduce the appropriate form of logistic recurrence relation, by finding values for the parameters r and E, in cases where linear expressions for the birth and death rates (hence also for the growth rate) are known. Normally, however, such expressions are not available in advance. This raises the question of how values for these parameters can be estimated directly from actual data about the population, in such a way that the corresponding logistic model 'fits the data' as closely as possible. As with the exponential model, we shall not go into full details about how this estimation is done, but the following example and activity indicate how the parameters for the logistic model may be estimated in some idealised cases.

Example 3.2 Revised model for gannets

For the population of gannets in Example 2.2 (page 22), it was shown that the proportionate growth rate was $r = 0.20$ at relatively low population levels. At higher population levels (later than 1919, see Figure 2.7(a)) this growth rate is no longer appropriate.

The size of this population was 3200 in 1924, 3600 in 1929 and 4000 in 1934. The growth was therefore 400 in each of the 5-year periods before and after 1929, when the population size was 3600. This suggests an annual growth rate of about 80 for a population size of 3600.

Assuming that the behaviour of this population satisfies a logistic recurrence relation, estimate the value of the equilibrium population level, E.

Solution

The logistic recurrence relation is

$$P_{n+1} - P_n = rP_n\left(1 - \frac{P_n}{E}\right), \tag{3.3}$$

where in this case $r = 0.20$ from consideration of the proportionate growth rate at low population levels. Since the annual growth rate $P_{n+1} - P_n$ is estimated to be 80 for the population size $P_n = 3600$, we have

$$80 = 0.20 \times 3600\left(1 - \frac{3600}{E}\right) = 720\left(1 - \frac{3600}{E}\right).$$

On solving this equation for E, we obtain

$$\frac{3600}{E} = 1 - \frac{80}{720} = \frac{8}{9}, \quad \text{from which} \quad E = 4050.$$

This is the equilibrium level for the population.

Activity 3.2 Revised model for the US population

For the US population (1790–1890), you found in Activity 2.2 (page 23) that the annual proportionate growth rate was $r = 0.028$. At higher population levels (later than 1900, see Figure 2.6(b)) this is no longer appropriate.

The size of the US population was 76 million in 1900, 92 million in 1910 and 106 million in 1920. Hence there was a growth of 30 million over a 20-year period with midpoint 1910, when the population was 92 million. This suggests an annual growth rate of about 1.5 million for a population size of 92 million.

Assuming that the behaviour of this population satisfies a logistic recurrence relation, estimate the value of the equilibrium population level, E. (As before, take the population size P_n to be measured in millions.)

A solution is given on page 56.

The approach used in the last example and activity can be applied only if the value of r is found first from data on the population growth at low population levels. When such data are not available, a strategy of the following type may be adopted instead.

Example 3.3 Using growth data at two population levels

A population of deer is introduced to a nature reserve on an island, and its growth is subsequently monitored. It is estimated that the annual proportionate birth rate is 2.70 when $P = 100$ and 2.40 when $P = 200$, while the annual proportionate death rate is 0.96 when $P = 100$ and 1.82 when $P = 200$. (In each case, P is the population size at the start of the year.)

(a) Assuming that the behaviour of this population satisfies the logistic model, estimate the values of the equilibrium population level, E, and proportionate growth rate for low population sizes, r.

(b) If the population size is 150 at the start of a particular year, what size of population is predicted for the start of the next year, according to the model?

Solution

(a) The proportionate growth rate is

$$2.70 - 0.96 = 1.74 \quad \text{when } P = 100,$$
$$2.40 - 1.82 = 0.58 \quad \text{when } P = 200.$$

Also, the proportionate growth rate has the form $r(1 - P/E)$ for a population size P (equation (3.2)). Hence we have two simultaneous equations involving r and E, namely,

$$1.74 = r\left(1 - \frac{100}{E}\right) \quad \text{and} \quad 0.58 = r\left(1 - \frac{200}{E}\right).$$

There are various ways of solving these equations. For example, on dividing through each equation by r and then rearranging, we obtain

$$\frac{1.74}{r} + \frac{100}{E} = 1 \quad \text{and} \quad \frac{0.58}{r} + \frac{200}{E} = 1,$$

You may find it helpful to express these equations in terms of $x = 1/r$ and $y = 1/E$.

which is a pair of simultaneous linear equations in the variables $1/r$ and $1/E$. We can eliminate the $1/E$ term by subtracting the second equation from twice the first equation:

$$\frac{3.48}{r} + \frac{200}{E} - \left(\frac{0.58}{r} + \frac{200}{E}\right) = 2 - 1,$$

which gives $2.9/r = 1$, that is, $r = 2.9$.

On substituting this value for r into the first equation, we obtain

$$\frac{1.74}{2.9} + \frac{100}{E} = 1; \quad \text{that is,} \quad \frac{100}{E} = 1 - \frac{1.74}{2.9} = \frac{1.16}{2.9} = 0.4.$$

Solving this equation gives $E = 250$.

(b) The logistic recurrence relation in this case is

$$P_{n+1} - P_n = 2.9P_n\left(1 - \frac{P_n}{250}\right).$$

If $P_n = 150$ for some n, then we have

$$P_{n+1} = 150 + 2.9 \times 150\left(1 - \frac{150}{250}\right) = 324.$$

(Note that, in this case, the population size has moved in one year from well below the equilibrium population level, 250, to well above it.)

Activity 3.3 Modelling beetles

Experiments are conducted on a species of beetle which breeds with clearly separated generations. In a series of experiments, different numbers of beetles are introduced into identical laboratory cultures, and the numbers in the next generation are counted. When the experiment is started with 100 beetles, the next generation numbers 220, whereas when the experiment is started with 400 beetles, the next generation numbers 160.

(a) Write down the proportionate growth rate, $(P_{n+1} - P_n)/P_n$, for $n = 0$, for each of $P_0 = 100$ and $P_0 = 400$.

Recall that P_n is the population size in the nth generation from the start of the experiment.

(b) Assuming that the behaviour of this population satisfies the logistic model, estimate the values of the equilibrium population level, E, and the proportionate growth rate for low population sizes, r.

(c) If the experiment is started with 200 beetles, how many will there be in the next generation, according to the model?

Solutions are given on page 56.

3.3 *Investigating the logistic model*

We have not, as yet, 'done the maths' to obtain closed-form solutions from the logistic model, beyond finding the two constant sequences, $P_n = E$ and $P_n = 0$. In fact, unlike the situation with the exponential model, there are only special cases for which closed-form solutions of the logistic recurrence relation,

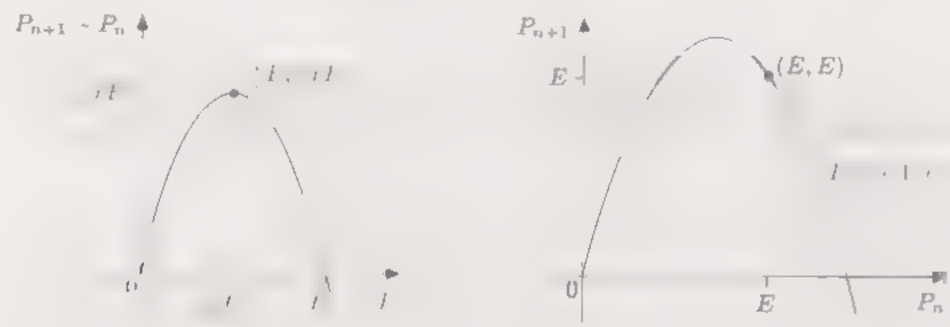
$$P_{n+1} - P_n = rP_n\left(1 - \frac{P_n}{E}\right), \tag{3.3}$$

are known. Study of the population behaviour predicted by this model therefore depends on either deductions from the form of the recurrence relation itself or direct calculation of successive terms using the recurrence relation. In this subsection we focus on what information can be obtained by looking at the model from the graphical and algebraic standpoints. In Section 4 we turn to direct calculation of terms, using the computer. Only once these matters have been explored can we pronounce upon how well the logistic model compares with reality.

Graphical approach

Essentially, this first way gives the graph in Figure 3.1, but with $P = 0$ excluded.

First, note that there are several ways in which the information given by equation (3.3) can be graphed. One way is to plot proportionate growth rate, $R(P_n) = (P_{n+1} - P_n)/P_n$, against P_n, which we illustrated when deriving the logistic recurrence relation. More useful, for current purposes, is a plot of growth rate, $P_{n+1} - P_n$, against P_n, which is shown in Figure 3.2(a). The graph is that of the function given by the right-hand side of equation (3.3), which is a quadratic function of P_n. It cuts the horizontal axis at $P_n = 0$ and $P_n = E$.



(a) (b)

Figure 3.2 Graphs from the logistic recurrence relation: (a) plot of $P_{n+1} - P_n$ against P_n, (b) plot of P_{n+1} against P_n

Note that population *growth* in a year means that $P_{n+1} - P_n > 0$, while population *decline* means that $P_{n+1} - P_n < 0$. Hence there is population growth where the graph of Figure 3.2(a) lies above the horizontal axis (that is, where $0 < P_n < E$) and population decline where the graph lies below this axis (where $P_n > E$). The equilibrium population level, $P_n = E$, appears as a point on the graph for which population growth is zero.

This maximum value is

$$r \times \tfrac{1}{2}E(1 - \tfrac{1}{2}) = \tfrac{1}{4}rE,$$

as marked on Figure 3.2(a).

Also, the growth rate will be a maximum at the vertex of the parabola, where $P_n = \frac{1}{2}E$. The further that P_n is from $\frac{1}{2}E$, the smaller is the corresponding growth rate

If the vertical axis were labelled by y, then we would call this a y-scaling.

Another point to note is that altering the value of the parameter r has the effect of scaling this parabola vertically, so that for a given value of P_n (and fixed E), the growth rate becomes larger in magnitude as r is increased.

The graph in Figure 3.2(b) is a plot of P_{n+1} (next year's population size) against P_n (this year's population size). This arises from rearranging equation (3.3) as follows:

$$\begin{aligned} P_{n+1} &= P_n + rP_n\left(1 - \frac{P_n}{E}\right) \\ &= P_n\left(1 + r\left(1 - \frac{P_n}{E}\right)\right) \end{aligned}$$

The right-hand side here is again a quadratic function of P_n, whose graph passes through the point (E, E). The graph cuts the horizontal axis when $P_{n+1} = 0$, that is, when $P_n = 0$ and when $1 + r(1 - P_n/E) = 0$. So the intercepts on the horizontal axis are at $P_n = 0$ and $P_n = E(1 + 1/r)$.

This latter value of P_n is of particular interest from the modelling point of view since, according to the graph, if $P_n = E(1 + 1/r)$ then $P_{n+1} = 0$, and if $P_n > E(1 + 1/r)$ then $P_{n+1} < 0$. Thus the population becomes extinct within a year, according to the model, if its numbers ever rise as high as $E(1 + 1/r)$. As the parameter r is increased, this 'ceiling on sustainability' becomes closer to the equilibrium population level, E.

Algebraic approach

We have already employed some algebra above, while investigating what useful information about the logistic model can be gleaned from graphs. The following argument, however, depends upon algebraic manipulation alone.

Starting once more from the logistic recurrence relation (3.3), we divide both sides of the equation by E, to obtain

$$\frac{P_{n+1}}{E} - \frac{P_n}{E} = r\frac{P_n}{E}\left(1 - \frac{P_n}{E}\right)$$

Now we put $x_n = P_n/E$, so that also $x_{n+1} = P_{n+1}/E$. This results in a recurrence relation for the new variable x_n, namely,

$$x_{n+1} - x_n = rx_n(1 - x_n), \tag{3.4}$$

which looks simpler than the original, since the parameter E no longer appears. Mathematically, it is a version of the logistic recurrence relation (3.3) with $E = 1$. In modelling terms, the quantity x_n represents the population size P_n as a proportion of its equilibrium level, E.

The derivation of equation (3.4) shows that the parameter E is just a scaling factor for any logistic recurrence sequence. So, in order to calculate terms of a sequence P_n which satisfies the logistic recurrence relation (3.3), with given values for the parameters r, E and for P_0, we could proceed as follows:

- calculate $x_0 = P_0/E$;
- calculate the required number of terms of the corresponding sequence x_n which satisfies the recurrence relation (3.4);
- calculate $P_n = x_n E$ for each relevant value of n.

The graph of P_n against n is just a vertically scaled version of the graph of x_n against n, with scaling factor E.

It follows that, in order to understand the long-term behaviour of sequences P_n which satisfy equation (3.3), for given r and E, we need to understand only the long-term behaviour of sequences x_n which satisfy equation (3.4). This observation saves considerable labour where computer investigation of equation (3.3) is concerned, since it tells us that there is no need to experiment with different values of the parameter E.

Origins of the logistic model

The first person to have formulated the logistic model is reputed to be the Belgian mathematician Pierre-François Verhulst (1804–49). He argued against the previous view, based on the ideas of Malthus, that populations tend to increase exponentially, by pointing out that there were countervailing forces which acted to slow growth down.

Based on his model, formulated in 1846, he predicted that the Belgian population would never exceed 9.4 million. Over 150 years later it is just above 10 million, but when compared with predictions from the exponential model, Verhulst's estimate looks remarkably good.

Summary of Section 3

This section has introduced:

- the logistic model for population variation, with recurrence relation

$$P_{n+1} = P_n + rP_n\left(1 - \frac{P_n}{E}\right),$$

  which assumes that proportionate growth rate is a decreasing linear function of population size;
- methods for estimating, from data about a population, values for the parameters r (proportionate growth rate at low population levels) and E (equilibrium population level);
- some graphical and algebraic analysis of the behaviour to be found according to the logistic model.

Exercises for Section 3

Exercise 3.1

UN data suggest that the proportionate growth rate *per decade* of the world population was 0.20 in 1960, when the population size was 3.02 billion, and 0.15 in 1990, when the population size was 5.27 billion.

(a) Assuming that the behaviour of the world population satisfies a logistic model, estimate the values of the equilibrium population level, E, and the parameter r.

(b) Using this model, and the world population estimate of 6.06 billion for 2000, what population is forecast in the year 2010

This is not a 'standard exercise'. It asks you to analyse algebraically the behaviour of the logistic model for population sizes close to the equilibrium level E, and provides predictions which can be compared with numerical results to be obtained in Section 4. However, if you are short of time then you may prefer to omit this exercise.

The new variable Q_n is (when positive) the amount by which the population size P_n is below its equilibrium value E.

Exercise 3.2

You saw at the end of Subsection 3.1 that where P_n/E is small compared with 1, the logistic model is approximated well by the exponential model. This exercise concerns a similar approximation for the case where $1 - P_n/E$ is small, that is, where P_n is close to its equilibrium level.

(a) Express the logistic recurrence relation,

$$P_{n+1} = P_n + rP_n\left(1 - \frac{P_n}{E}\right),$$

in terms of Q_n, where $Q_n = E - P_n$.

(b) Show that if Q_n/E is small compared with 1, then Q_n satisfies approximately the geometric recurrence relation

$$Q_{n+1} = (1 - r)Q_n.$$

(c) Use this result to describe the nature of logistic recurrence sequences with terms close to $P_n = E$, in each of the following cases.

(i) $0 < r < 1$ (ii) $1 < r < 2$ (iii) $r > 2$

4 Logistic recurrence sequences on the computer

To study this section you will need access to your computer, together with Computer Book B.

Bearing in mind the qualitative information gleaned in Subsection 3.3 about the logistic model, it is now time to calculate terms of logistic recurrence sequences directly via the computer. The aim is to try and build up an overview of such sequences, and to see how their behaviour depends on the value of the parameter r. (As demonstrated in Subsection 3.3, there is no need to consider separately the effect of varying the value of E.)

In Example 3.1 you saw that the growth in a certain barnacle goose population can be modelled by a logistic recurrence relation with $r = 0.20$ and $E = 13\,300$. Figure 4.1 shows the graph of a sequence generated by this recurrence relation, but with $P_0 = 20$ rather than the value of 3200 quoted in Example 3.1. This graph, which is S-shaped overall, shows that the values of P_n initially increase more and more rapidly. As you saw in Subsection 3.1, this initial growth is similar to that of a geometric sequence, $P_n = (1 + r)^n P_0$. The rate of growth reaches a maximum at $P_n = \frac{1}{2}E = 6650$, as forecast earlier with reference to Figure 3.2(a), then tails off as $P_n = E$ is approached. In the long term, values of P_n settle down close to $E = 13\,300$ and become effectively constant.

The corresponding picture for $P_0 = 3200$ is obtained by moving the vertical axis in Figure 4.1 to the right, until the intercept of the curve on it is 3200. The position of the vertical axis then corresponds

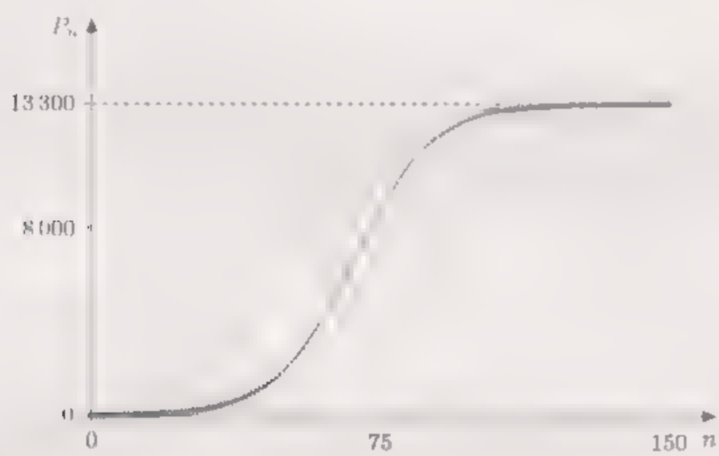


Figure 4.1 Graph of logistic recurrence sequence with $r = 0.20$, $E = 13\,300$, $P_0 = 20$

Are the features that are apparent here shared by sequences also generated by the logistic recurrence relation but with other values of r? You can now use your computer to examine this question.

Refer to Computer Book B for the computer-based work in this section.

Reviewing the results

These various forms of behaviour may also be seen in sequences generated by other non linear recurrence relations.

Computer investigation of logistic recurrence sequences reveals an interesting variety of behaviour as the parameter r is increased. The apparently unstructured behaviour shown by sequences when $r = 2.75$ and $r = 3$ is referred to as **chaotic**. The sequence behaviour of repeatedly taking a number of different but repeating values, seen with $r = 2.25$ or $r = 2.5$, for example, is called **cycling**. For $r = 2.25$, the sequence P_n has a **2-cycle**, and for $r = 2.5$, P_n has a **4-cycle**.

You may use this table for reference in future work (see Exercise 5.1), including assignments.

The table below provides a summary of information on the extent of various behaviour regimes for logistic recurrence sequences, as may be inferred from computer investigation.

Range of r	Long-term behaviour of P_n
$0 < r \leq 1$	Settles close to (converges to) E, with values always just below E
$1 < r \leq 2$	Settles close to E, with values alternating between just above and just below E
$2 < r \leq 2.44$	2-cycle, with one value above E and one value below E
$2.45 \leq r \leq 2.54$	4-cycle, with two values above E and two values below E
$2.6 \leq r \leq 3$	Chaotic variation between bounds (with some exceptions)

Evaluating the logistic model

The remainder of this section will not be assessed

In Subsection 3.1 we constructed a model of populations based on the logistic recurrence relation. We shall now complete the modelling cycle by briefly evaluating this model, with reference to the examples of actual population variation given in various figures of Section 2.

Many of the broad features of sequences generated by the logistic recurrence relation do correspond to aspects of population examples seen earlier.

- For positive values of r less than 1, the sequence generated by the logistic recurrence relation exhibits an S-shaped increase, provided that the choice of P_0 is small compared with E. Such a pattern is shown in Figure 2.7(b). The examples in Figures 2.6 (US population), 2.7(a) and 2.7(c) also match the general form of *part* of the S-shape. The graphs in Figures 2.6 and 2.7(a) have not yet approached close to E, while Figure 2.7(c) starts with a value of P_0 that is not small relative to E, so the initial 'geometric increase' part of the graph is missing. This is also the case with the graph for grey herons in Figure 2.5(b), from 1963 onwards.
- For $r = 1.8$, the sequence generated by the logistic recurrence relation shows rapid increase, followed by oscillations, settling to an equilibrium value. This is the same pattern as shown by the population of beetles in Figure 2.8(a).
- The graph for the lemming population in Figure 2.8(b) shows systematic cycles, which might reflect the cyclic behaviour of sequences generated by the logistic recurrence relation for values of r in the range between 2 and 2.54.

- There are populations which show wide and unsystematic variation. This could correspond to the chaotic behaviour of the sequences generated for many values of r between 2.6 and 3.
- As a final specific comparison, the graph for the pheasants in Figure 2.1 (for either April or October data) shows rapid increase This could match the initial part of an S-shaped curve (for $r < 1$), or resemble the rapid initial growth of a sequence generated for a larger value of r.

From the result of Activity 3.1 the latter is more likely.

Sequences generated by the logistic recurrence relation (with $r < 3$) can be seen to correspond in their general features to several different patterns of variation shown by actual animal populations. However, matching specific numerical predictions based on the logistic model of populations to real data shows the model to be less reliable. The comparison with reality shows a qualitative rather than a quantitative match, in many cases. This is hardly surprising, given the rather sweeping assumptions on which the model is based. We have assumed that the environmental conditions are unchanging, that there is no migration, and that the proportionate growth rate (birth rate minus death rate) decreases *linearly* as the population size increases. It is unlikely in any real case that all of these conditions will be met closely. The assumption about the growth rate was based on simplicity, rather than on any underlying biological reason, and will not hold precisely in practice.

We have looked at only positive values for r. The logistic model can be used, for $P_0 < E$, with negative values of r, but it then has little to offer which is not provided by the exponential model of decline.

One particular criticism of the growth rate assumption is that it predicts extinction within a year for a population whose current size exceeds $E(1 + 1/r)$, and this is probably unrealistic. A population suffering from excessive overcrowding would be expected to survive, albeit at a much lower level. Other models have been put forward to overcome this drawback. One of these is the *Ricker model*, based on the recurrence relation

See Figure 3.2(b).

$$P_{n+1} = (1 + r)P_n \exp(-aP_n),$$

where r has the same interpretation as in the logistic model, a is a positive parameter and exp is the exponential function (introduced in Chapter A3, Subsection 3.2). The graphs of P_{n+1} against P_n which can be drawn for this model (the analogues of graphs such as that in Figure 3.2(b)) are known as *Ricker curves*, and are much used to describe the behaviour of fish populations and to determine sustainable fishing levels.

William Ricker, born in 1908, was until retirement Chief Scientist to the Fisheries Board of Canada. He developed this model in the 1950s.

Summary of Section 4

In this section you investigated sequences generated by the logistic recurrence relation, $P_{n+1} - P_n = rP_n(1 - P_n/E)$, via the direct evaluation of terms by computer. The choice of starting value P_0 does not usually affect the long-term behaviour of the sequences. This behaviour alters qualitatively as the parameter r is increased, as summarised in the table on page 40. Some sequences have 2-cycles, 4-cycles or chaotic behaviour in the long term.

5 Sequences and limits

In this section we address some issues which arose in the context of investigating logistic recurrence sequences, though they have much wider application. A sequence of numbers, u_n ($n = 1, 2, 3, \ldots$), may or may not 'settle down' closer and closer to a single number in the long term (as n becomes larger and larger). A sequence which does settle down in this way is said to be *convergent*, and the number which it approaches more and more closely is called the *limit* of the sequence. We are concerned here with these ideas of convergence and limits for sequences.

Subsection 5.1 looks at what can be said in this regard about sequences defined by recurrence systems, while Subsection 5.2 turns to sequences described by closed-form formulas. Subsection 5.3 focuses on the topic of sequences whose terms are the sums of terms of other sequences. These are the *series* of which you saw examples in Subsection 1.2.

5.1 Sequences from recurrence systems

Suppose that P_n is a sequence generated by the logistic recurrence relation

$$P_{n+1} - P_n = rP_n\left(1 - \frac{P_n}{E}\right). \tag{3.3}$$

In Section 4 you saw that, in certain cases, the sequence P_n settles down in the long term to values that are effectively constant. When this happens, we say that the sequence P_n is **convergent** or that it **converges**. The value near which P_n settles in the long term is called the **limit** of the sequence. For example, with $P_0 = 3200$, $E = 1300$ and $r = 0.20$, the sequence P_n converges to the limit 13 300, as indicated by the graph in Figure 5.1(a). This convergence is described concisely by the notation

If a sequence has a limit, then that limit is unique.

$$\lim_{n\to\infty} P_n = 13\,300,$$

or, equivalently, by

$$P_n \to 13\,300 \text{ as } n \to \infty,$$

a notation introduced in Chapter A1. (Some books print the limit as $\lim_{n\to\infty} P_n$ when it appears in a line of text.) However, with the same values of P_0 and E but with $r = 2.9$, P_n is *not* convergent, since it does not settle near any particular value in the long term; see Figure 5.1(b).

A sequence which is not convergent is said to be **divergent**.

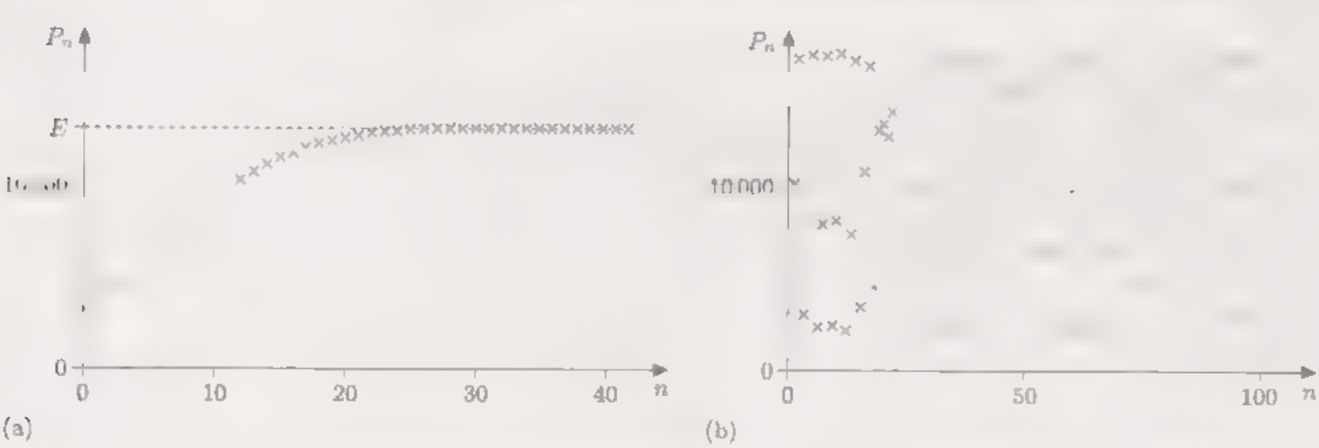


Figure 5.1 Graphs of logistic recurrence sequences with $P_0 = 3200$ and $E = 13\,300$ for (a) $r = 0.20$, (b) $r = 2.9$

You saw in Subsection 3.1 that if the logistic recurrence relation generates a constant sequence, $P_n = c$ say, then there are only two values that c can take, namely 0 and E. If a sequence generated by the logistic recurrence relation does converge, then for large enough n the terms of the sequence are effectively equal. Thus in the long term, a convergent sequence is effectively a constant sequence, so its limit must be one or other of the values 0 or E. However, this says only that convergence to any other limit is impossible. As the case above with $r = 2.9$ shows, there is no guarantee of convergence even when potential limit values have been identified.

The above method of finding possible limit values applies to other types of sequence generated by recurrence relations.

Example 5.1 Finding possible limit values for a sequence

Suppose that a sequence x_n is generated by the recurrence relation

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right) \quad (n = 0, 1, 2, \ldots). \tag{5.1}$$

Each non-zero value of x_0 here will give a different sequence.

To what limit values might such a sequence x_n converge?

Solution

Suppose that $x_n = c$ is a constant sequence generated by the given recurrence relation. Then we have $x_{n+1} = x_n = c$, and so

$$c = \tfrac{1}{2}(c + 2/c).$$

This is equivalent to

$$2c = c + 2/c; \quad \text{that is,} \quad c = 2/c.$$

So we obtain $c^2 = 2$ and hence $c = \pm\sqrt{2}$. These are the two possible limit values for any sequence generated by the recurrence relation (5.1).

Activity 5.1 Finding possible limit values for a sequence

A population of ravens on an island has size 50 at the start of year 0. Their proportionate birth and death rates are constant, being respectively 0.3 and 0.4. There is no emigration from the island, but 8 ravens join the population from outside just before the start of each year.

(a) Show that the variation in this population can be modelled by the recurrence system

$$P_0 = 50, \quad P_{n+1} = 0.9P_n + 8 \quad (n = 0, 1, 2, \ldots), \tag{5.2}$$

where P_n is the population size at the start of year n.

(b) Find any possible equilibrium level for this population. (This is equivalent to seeking a value c for which the constant sequence $P_n = c$ is a solution to the recurrence relation.)

Solutions are given on page 57.

If a sequence is specified by a recurrence system, then the method of Example 5.1 and Activity 5.1(b) may tell us the possible limit values to which the sequence can converge. This is useful but limited information, since there is no accompanying indication of whether or not convergence to a possible limit actually occurs.

In general, sequences may be specified through a recurrence system or through a closed form. Where available, a closed form can offer greater opportunities for investigating whether or not a sequence converges, as you will see in the next subsection.

In other cases, no closed form is available. This is true of any non-constant sequence generated by the recurrence relation (5.1). Despite this, if x_0 is not equal to zero, then any such sequence will converge, to the limit $\sqrt{2}$ if x_0 is positive and to the limit $-\sqrt{2}$ if x_0 is negative. In terms of the 'lim' notation introduced earlier, this may be expressed as

$$\lim_{n\to\infty} x_n = -\sqrt{2} \quad \text{if } x_0 < 0, \qquad \lim_{n\to\infty} x_n = \sqrt{2} \quad \text{if } x_0 > 0.$$

5.2 Sequences from closed-form formulas

Using the modulus

The modulus of a real number was defined in Chapter A3, Subsection 1.2.

In this subsection, the modulus of a real number is used to permit concise expression of certain types of inequality and to express the distance from the origin to a point on the number line.

The distances from the origin to the points 5 and -4 on the number line are 5 and 4, respectively. Since the modulus of a real number x is defined by

$$[illegible] \begin{cases} x, & \text{if } x \geq 0, \\ [illegible] & \text{if } [illegible] \end{cases}$$

we have

$$|5| = 5 \quad \text{and} \quad |-4| = 4,$$

as shown in Figure 5.2.

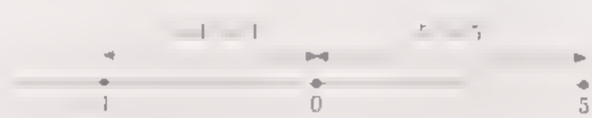


Figure 5.2 Distances from the origin

More generally, the distance from any point p to another point q is given by $|p-q|$.

In general, the **distance** from the origin to a point a on the number line is given by $|a|$.

Consider the inequalities $-1 < a < 1$ (that is, $-1 < a$ and $a < 1$). These say that the distance from a to the origin is less than 1. Thus

$$-1 < a < 1 \quad \text{and} \quad |a| < 1$$

are equivalent inequalities.

Manipulating closed forms

In Activity 5.1(b) you found a single equilibrium population value (80) for an island colony of ravens whose population variation satisfies the recurrence system (5.2). This leaves open the question of whether the size of this population will actually converge to its equilibrium level.

Since the recurrence relation here is *linear*, it is possible to find a closed-form formula for P_n, which is

$$P_n = 80 - 30(0.9)^n \quad (n = 0, 1, 2, \ldots).$$

The closed-form formula for a linear recurrence system was given in Chapter A1, Section 4.

Given this formula, it is possible to deduce the long-term behaviour of the population sequence P_n without any need for calculation. This is because if n is large, then $(0.9)^n$ is small (that is, close to 0). With n sufficiently large, $30(0.9)^n$ is also small, and we have $30(0.9)^n \to 0$ as $n \to \infty$. According to the closed-form formula for P_n, therefore, the sequence converges, to the limit 80, which is the equilibrium level. Thus the raven population tends to this equilibrium level. A graph of this sequence is shown in Figure 5.3.

The long-term behaviour of r^n, for different values of r, was the subject of Chapter A1, Subsection 5.2. In particular, if $|r| < 1$, then

$$r^n \to 0 \text{ as } n \to \infty.$$

Here we have $r = 0.9$.

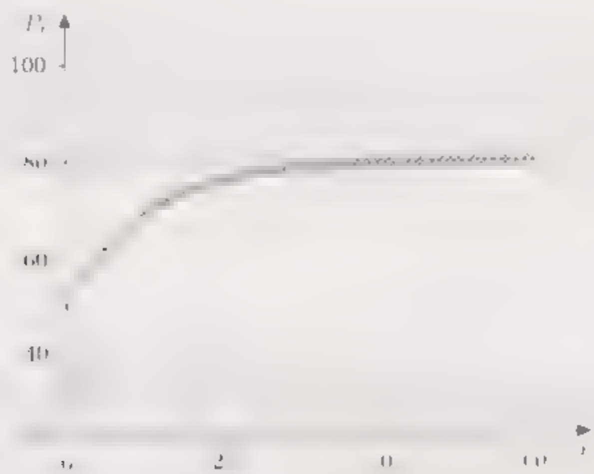


Figure 5.3 Graph of $P_n = 80 - 30(0.9)^n$

Notice how the graph of the sequence P_n settles near a horizontal straight line as n becomes large. This behaviour corresponds to convergence of the sequence, with the horizontal line being at the limit value.

A similar approach enables you to ascertain the long-term behaviour of some other sequences defined by closed-form formulas. Not every such sequence is convergent. For example, the sequences

$$a_n = n^2 \quad \text{and} \quad a_n = 2^n$$

Assume, here and in the discussion below, that $n = 1, 2, 3, \ldots$.

both have values that increase without limit as the value of n increases. On the other hand, the values of $a_n = -5n$ *decrease* as n increases, and become arbitrarily large while remaining negative. In saying that the terms of a sequence a_n 'become arbitrarily large', we mean that the terms become arbitrarily far from 0. This description can be expressed in terms of the distance $|a_n|$: 'a_n becomes arbitrarily large' means that $|a_n| \to \infty$ as $n \to \infty$. This is true of the sequence $a_n = -5n$, for which we also have $a_n \to -\infty$ as $n \to \infty$. Thus none of the sequences

As pointed out in Chapter A1, Subsection 5.2, such sequences are also called *unbounded*.

$$a_n = n^2, \quad a_n = 2^n, \quad a_n = -5n \tag{5.3}$$

is convergent. The same applies for any other sequence defined by a closed form where the terms, whether positive or negative, become arbitrarily large in the long run.

There are other closed forms for sequences a_n where you can see that a_n becomes smaller and smaller (arbitrarily small, that is, arbitrarily close to 0) as n becomes large. Examples of this are

$$b_n = \frac{1}{n^2}, \quad b_n = \frac{1}{2^n} \quad \text{and} \quad b_n = -\frac{1}{5n}.$$

These sequences *are* convergent, with limit 0. In each case, b_n is of the form $1/a_n$, where terms of the sequence a_n (given respectively by equations (5.3)) become arbitrarily large as n increases. This highlights a general rule to use when considering the convergence of sequences.

Reciprocal Rule

If the terms of a sequence b_n are of the form $1/a_n$, where terms of the sequence a_n become arbitrarily large as n increases, then the sequence b_n is convergent, and $\lim_{n\to\infty} b_n = 0$.

Another useful result concerns what happens to a sequence in the long run if each of its terms is multiplied by the same number. For example, it has already been remarked that the sequence $a_n = 1/n^2$ converges to 0, since its values eventually become arbitrarily small as n increases. Multiplying all terms a_n by 50 makes them all larger by this factor, but this does not prevent the resulting sequence, $b_n = 50/n^2$, becoming arbitrarily small in the long run.

This rule was used informally in Chapter A1, Subsection 5.2.

Constant Multiple Rule

If the terms of a sequence b_n are of the form ca_n, where the sequence a_n is convergent with limit 0, and c is a constant, then the sequence b_n is also convergent, and $\lim_{n\to\infty} b_n = 0$.

These two rules may be used in combination. For example, the sequence $3 + n$ becomes arbitrarily large as n increases, so (by the Reciprocal Rule) the sequence $1/(3 + n)$ converges to the limit 0. Thus (by the Constant Multiple Rule) the sequence $100/(3 + n)$ also converges to the limit 0.

In more complicated formulas, you can look for quantities that become small as n becomes large. For example, consider the sequence

$$a_n = \frac{5000}{1 + 24(0.5)^n}. \tag{5.4}$$

Since $0.5 < 1$, the quantity $(0.5)^n$ becomes arbitrarily small as n becomes large. The same is true of $24(0.5)^n$ (by the Constant Multiple Rule), so $1 + 24(0.5)^n$ tends to 1. In the long run, the values of a_n become arbitrarily close to 5000. In other words, the sequence defined by equation (5.4) converges to the limit 5000.

In arguments such as this, it is useful to recall the known behaviour for large n of the 'basic sequences' r^n and n^p.

Long-term 'basic sequence' behaviour

The long-term behaviour of the sequence r^n $(n = 1, 2, 3, \ldots)$ is as follows.

- If $|r| > 1$, then $|r^n| \to \infty$ as $n \to \infty$. (If $r > 1$ then $r^n \to \infty$, whereas if $r < -1$ then r^n is unbounded and alternates in sign.)
- If $|r| < 1$, then $r^n \to 0$ as $n \to \infty$.
- If $r = 1$, then $r^n = 1$. If $r = -1$, then r^n alternates between 1 and -1.

The long-term behaviour of the sequence n^p $(n = 1, 2, 3, \ldots)$ is as follows.

- If $p > 0$, then $n^p \to \infty$ as $n \to \infty$.
- If $p < 0$, then $n^p \to 0$ as $n \to \infty$.
- If $p = 0$, then $n^p = 1$.

The statements here about r^n follow from a table in Chapter A1, Subsection 5.2.

For example, if $p = 3$ then we have

$$n^3 \to \infty \text{ as } n \to \infty,$$

whereas if $p = -2$ then, since $n^{-2} = 1/n^2$, we have

$$n^{-2} \to 0 \text{ as } n \to \infty.$$

In the next example and activity, you are asked to decide whether or not various sequences defined by closed-form formulas converge and, if they do, to what limits. Reasoning of the type above should pay dividends here.

If you find this difficult at first, then there is always the option of working out some numerical values of terms (for $n = 1$, $n = 10$, $n = 100$, perhaps, using a calculator where necessary) in order to try and spot what is going on.

Example 5.2 Long-term behaviour from closed-form formulas

For each of the sequences below, decide whether or not it converges and, if it does, to what limit. In each case $n = 1, 2, 3, \ldots$.

(a) $a_n = (0.6)^n + 2(1.4)^n$ (b) $a_n = \dfrac{5}{17 - 3(2.1)^n}$ (c) $a_n = \dfrac{n^2}{5 + 10n}$

Solution

(a) For large n, the sequence $(0.6)^n$ converges to 0 (it is r^n with $r = 0.6 < 1$), but the sequence $(1.4)^n$ (which is r^n with $r = 1.4 > 1$) is unbounded. Hence the sequence a_n does not converge.

(b) For large n, the sequence $(2.1)^n$ is unbounded, so the same is true of $17 - 3(2.1)^n$. Hence, by the Reciprocal Rule, $1/(17 - 3(2.1)^n)$ converges to 0, so by the Constant Multiple Rule (with $c = 5$), the sequence a_n converges to the limit 0; that is, $\lim_{n \to \infty} a_n = 0$.

(c) The given sequence can be written (dividing top and bottom of the fraction by n) as

$$a_n = \frac{n}{5/n + 10}.$$

Now $5/n$ converges to 0 for large n ($1/n$ is of the form n^p with $p = -1$), so $5/n + 10$ converges to 10. Hence, for large n, the sequence a_n will behave like $\frac{1}{10}n$, which is not convergent. We conclude that a_n itself is not convergent either.

Comment

In part (c), since neither n nor n^2 converges, it would appear that a_n is the ratio of two sequences that do not converge. From this fact alone, we can make no deductions about the convergence of a_n. To make progress, we rewrite a_n in the form given, to produce a sequence in the denominator that does converge.

Activity 5.2 Long-term behaviour from closed-form formulas

For each of the sequences below, decide whether or not it converges and, if it does, to what limit. In each case $n = 1, 2, 3, \ldots$.

(a) $a_n = \dfrac{1}{2 + 3n}$

(b) $a_n = 5 + n^3$

(c) $a_n = \dfrac{100}{4 + 20(0.6)^n}$

(d) $a_n = 3 + (-1)^n$

(e) $a_n = \dfrac{60n}{3 + 5n}$

Solutions are given on page 57.

Courses on analysis use more formal approaches to establish the long-term behaviour of sequences. For example, the sequence $P_n = 80 - 30(0.9)^n$ would be dealt with by identifying $80 - 30(0.9)^n$ as the sum of two sequences. This approach requires that 80 is thought of as the constant sequence $a_n = 80$, which converges to 80, and an appeal to a sum rule as well as the Constant Multiple Rule.

5.3 Sequences from sums

You first saw this in Chapter A1, Subsection 4.2.

In Example 1.1(c), we recalled the formula for the sum of a finite geometric series. Using sigma notation, this is

$$\sum_{i=0}^{n} ar^i = a\left(\frac{1 - r^{n+1}}{1 - r}\right) \quad (r \neq 1). \tag{5.5}$$

Now the terms ar^i ($i = 0, 1, 2, \ldots$) form a geometric sequence. From this sequence, it is possible to construct a new sequence s_n, each term of which is a sum of consecutive terms from the sequence ar^i. Thus we have

$$s_0 = a, \quad s_1 = a + ar, \quad s_2 = a + ar + ar^2,$$

and so on. The nth term of the new sequence is given by

$$s_n = \sum_{i=0}^{n} ar^i \quad (n = 0, 1, 2, \ldots),$$

which is the sum of a finite geometric series. Does the new sequence s_n have a limit? If it does, then $\lim\limits_{n\to\infty} s_n$ exists and we write

$$\lim_{n\to\infty} s_n = \sum_{i=0}^{\infty} ar^i.$$

This limit is the sum of an *infinite* geometric series,

$$a + ar + ar^2 + ar^3 + \cdots$$

Thanks to equation (5.5), it is possible to say almost immediately when such a sum exists and, when it does, what its value is. If $|r| > 1$, then the quantity r^{n+1} is unbounded for large n, so the same is true of s_n. If $r = 1$, then we have

$$s_n = \underbrace{a + a + a + \cdots + a}_{n+1 \text{ such terms}} = (n+1)a,$$

which is unbounded for large n. If $r = -1$, then the terms of s_n take the alternate values a and 0. Hence convergence can occur only if $|r| < 1$. In this case, as recalled in Subsection 5.2, $\lim\limits_{n\to\infty} r^n = 0$. From equation (5.5), it follows that

$$\sum_{i=0}^{\infty} ar^i = \lim_{n\to\infty} a\left(\frac{1 - r^{n+1}}{1-r}\right) = \frac{a}{1-r} \quad (|r| < 1). \tag{5.6}$$

The sequence r^{n+1} is the sequence r^n without the first term r^0, so

$$\lim_{n\to\infty} r^{n+1} = \lim_{n\to\infty} r^n.$$

Example 5.3 Sum of an infinite geometric series

What is the sum of the following infinite series?

$$\tfrac{1}{2} + (\tfrac{1}{2})^2 + (\tfrac{1}{2})^3 + \cdots$$

Solution

This is an infinite geometric series, with $a = \frac{1}{2}$ and $r = \frac{1}{2}$, so by equation (5.6), the sum is

$$\sum_{i=0}^{\infty} \tfrac{1}{2}(\tfrac{1}{2})^i = \frac{\frac{1}{2}}{1 - \frac{1}{2}} =$$ [illegible]

Figure 5.4 shows a geometric interpretation of this result.

Figure 5.4 Illustration of an infinite geometric series with finite sum

Activity 5.3 Sum of an infinite geometric series

What is the sum of the following infinite series?

1 [illegible]

A solution is given on page 57.

We derived equation (5.6) by taking the limit for large n of equation (5.5), for $|r| < 1$. An alternative approach to summing an infinite geometric series comes in handy when the series concerned is in the form of a recurring decimal. This approach permits us to find the fraction which is equivalent to any recurring decimal, as the next example illustrates.

Note that this decimal can be considered as an infinite geometric series, with $a = 123 \times 10^{-3}$ and $r = 10^{-3}$. Recognition of these values, followed by direct application of equation (5.6), is not the quickest way to find the equivalent fraction, though you can check here that equation (5.6) gives the same result.

Example 5.4 Fraction equivalent to recurring decimal

Find the fraction equivalent to $0.123\,123\,123\ldots$.

Solution

Put $s = 0.123\,123\,123\ldots$. The repeating group, '123', is 3 digits long, so we multiply s by 10^3, to obtain

$$1000s = 123.123\,123\ldots = 123 + s.$$

Hence we have

$$s = \frac{123}{999} = \frac{41}{333}.$$

Activity 5.4 Fraction equivalent to recurring decimal

Find the fraction equivalent to $0.454\,545\ldots$.

A solution is given on page 57.

5.4 Some further sequences and series

This subsection will not be assessed.

As you saw in Subsection 5.2, it is sometimes possible to tell from a closed-form formula whether or not the corresponding sequence is convergent. However, this is not always the case. For example, is the sequence

$$a_n = \left(1 + \frac{1}{n}\right)^n \quad (n = 1, 2, 3, \ldots) \tag{5.7}$$

convergent? In the long term, there is an interplay between the quantity $1 + 1/n$, approaching 1 more and more closely, and the fact that this quantity (always greater than 1) is raised to the power n. The combined effect is not easy to predict. In such cases, as with sequences given by recurrence systems, calculation of enough terms of the sequence, perhaps using a computer, may help to indicate whether or not the sequence converges. Applying this approach to the sequence (5.7) gives the graph in Figure 5.5, which does appear to indicate that the sequence converges. Although the value of the limit cannot be read off accurately from the graph, it turns out that it is the base for natural logarithms, $e = 2.718\,281\ldots$.

This limit is established in courses on real analysis. The natural logarithm function ln, with base e, was introduced in Chapter A3, Subsection 4.3

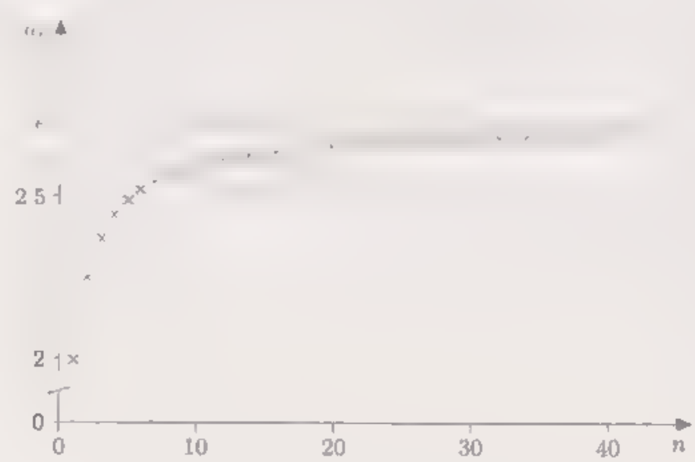


Figure 5.5 Approaching e

In Subsection 5.3 you saw how an infinite geometric series can be summed, provided that the common ratio is less than 1 in magnitude. Some infinite series which are not geometric can also be summed, and many fascinating formulas arise in this way; for example,

$$\sum_{i=0}^{\infty} \frac{(-1)^i}{2i+1} = 1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \cdots = \tfrac{1}{4}\pi$$

$$\sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{i} = 1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \cdots = \ln 2$$

If a_i $(i = 1, 2, 3, \ldots)$ is a sequence, then the sum $\sum_{i=1}^{\infty} a_i$ can exist only if a_i becomes smaller and smaller in the long term, that is, only if $\lim_{i \to \infty} a_i = 0$. However, this requirement of convergence to 0 for the sequence a_i is not on its own sufficient to guarantee the existence of a sum for the infinite series. For example, the sequence $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ converges to 0 but, as indicated below, there is no sum for the infinite series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$.

A non-existent sum

The following brief argument (by contradiction) shows that the infinite series

$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \cdots$$

has no sum.

Suppose that the sum of the infinite series exists and is equal to s; that is,

$$s = 1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \cdots.$$

Multiply both sides by $\frac{1}{2}$, to obtain

$$\tfrac{1}{2}s = \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{6} + \tfrac{1}{8} + \cdots. \tag{5.8}$$

These manipulations of infinite series can be shown to be valid if the series have sums.

The right-hand side is a *subseries* of the original. On subtracting it from the original, we have

$$\tfrac{1}{2}s = 1 + \tfrac{1}{3} + \tfrac{1}{5} + \tfrac{1}{7} + \cdots. \tag{5.9}$$

But now each term in series (5.9) is larger than its counterpart in series (5.8): $1 > \frac{1}{2}$, $\frac{1}{3} > \frac{1}{4}$, and so on. Hence the sum of series (5.9) must be greater than that of series (5.8) – but each is equal to $\frac{1}{2}s$! This contradiction shows that the original series has no sum.

Summary of Section 5

This section has introduced:

- the concepts of convergence and limit for sequences, with the notation $\lim_{n \to \infty} a_n$ for the limit of a convergent sequence a_n;
- for sequences defined by recurrence systems, the method of identifying potential limit values by finding all constant sequences that satisfy the recurrence relation;
- for sequences a_n defined by closed-form formulas, methods of reasoning based on how individual parts of the formula behave as n becomes large;
- the Reciprocal Rule and Constant Multiple Rule for sequences;

- the behaviour for large n of the 'basic sequences' r^n and n^p, as summarised in the box on page 47;
- a formula $(a/(1-r))$ for the sum of any infinite geometric series, $\sum_{i=0}^{\infty} ar^i$, with $|r| < 1$;
- a method for finding the fraction equivalent to a given recurring decimal.

Exercises for Section 5

Exercise 5.1

Consider the sequences defined by the logistic recurrence relation (3.3) with the parameters stated below. Basing your answers on the investigations in Section 4, decide in each case whether or not the sequence converges and, if it does, to what limit. Also, for each case where the sequence converges, state whether P_n ever exceeds E.

(a) $r = 1.8$, $E = 200$, $P_0 = 10$

(b) $r = 2.25$, $E = 1500$, $P_0 = 8$

(c) $r = 0.4$, $E = 1000$, $P_0 = 10$

Exercise 5.2

The following sequences are given by closed-form formulas. In each case, decide whether or not the sequence converges and, if it does, find the limit. In each case, $n = 1, 2, 3, \ldots$.

(a) $a_n = 4 + (-0.5)^n$ (b) $a_n = n^2 + 1/n$ (c) $a_n =$ [illegible]

Exercise 5.3

The following sequences are given by recurrence systems. In each case, use the method of Example 5.1 and Activity 5.1(b) to find all possible limit values for the sequence. Then use methods from Chapter A1 to find a closed form for the sequence, and use this to determine the long-term behaviour of the sequence. In each case, $n = 0, 1, 2, \ldots$.

(a) $x_0 = 3$, $x_{n+1} = 2x_n$

(b) $x_0 = 3$, $x_{n+1} = 0.2x_n$

(c) $x_0 = 3$, $x_{n+1} = 0.3x_n + 140$

Exercise 5.4

What is the sum of the following infinite series?

$$1 + \text{[illegible]} + \cdots$$

Exercise 5.5

Find the fraction equivalent to $0.513\,513\,513\ldots$.

Summary of Chapter B1

In this chapter you have seen sequences put to use in the development of models for the economics of car ownership and for variations in animal populations. In the context of the logistic model you saw that, while numerical investigation has its uses, prior reasoning may profitably reduce the scope of the task.

Logistic recurrence sequences exhibit an interesting variety of long-term behaviour. The topic of what happens to sequences in the long term was studied further from the mathematical point of view, with reference to convergence and limits. These concepts may be applied both to sequences and to series. The sum of either a finite or an infinite series may be expressed concisely using sigma notation.

Learning outcomes

You have been working towards the following learning outcomes.

Terms to know and use

Sigma (sum) notation, series, proportionate growth rate, exponential (geometric, Malthusian) model for populations, logistic model for populations, logistic recurrence relation, equilibrium population level, chaotic behaviour (of sequences), cycling, 2-cycle, 4-cycle, long-term behaviour (of sequences), convergent sequence, converges, limit of a sequence, arbitrarily large/small/close to, Reciprocal Rule and Constant Multiple Rule for sequences, infinite (geometric) series.

Symbols and notation to know and use

$\sum_{i=1}^{n} a_i$, $\lim_{n \to \infty} a_n$, $\sum_{i=1}^{\infty} a_i$.

Modelling skills

- ◇ Appreciate the roles of the various stages of the modelling cycle, as summarised in Figure 1.1.
- ◇ Interpret mathematical information obtained from a model in terms of the original purpose or situation.
- ◇ Evaluate a model in the light of what it predicts and how this compares with the real situation being modelled.
- ◇ Be aware of the possible need to revise a model on the basis of evaluation of it.

Mathematical skills

- Find the parameter values for the logistic model which correspond to given information about the proportionate birth and death rates (or growth rate) of a population.
- Obtain information about a sequence generated by the logistic recurrence relation, for given values of its parameters.
- Recognise various possible forms of long-term behaviour for sequences, in particular: convergence to a limit; cycling; chaotic behaviour; terms which become arbitrarily large.
- Apply and manipulate sigma notation, where appropriate.
- Apply the formula $\sum_{i=1}^{n} i = \frac{1}{2}n(n+1)$, where appropriate.
- Apply, where appropriate, either of the formulas

$$\sum_{i=0}^{n} ar^i = a\left(\frac{1-r^{n+1}}{1-r}\right) \quad (r \neq 1),$$

$$\sum_{i=0}^{\infty} ar^i = \frac{a}{1-r} \quad (|r| < 1).$$

- Find the fraction which is equivalent to a given recurring decimal.

Mathcad skills

- Calculate and graph terms of a logistic recurrence sequence, given a starting value and values for the parameters.
- Interpret the outcomes obtained from such calculations, in particular as regards the long-term behaviour of the sequence.

Ideas to be aware of

- The principle of simplicity (in mathematical modelling).
- That the prior application of graphical or algebraic reasoning may reduce the scope needed for a numerical investigation.

Solutions to Activities

Solution 1.1

You might, for example, consider:

- ◇ whether to buy new or used if used, how old, and whether to buy from a dealer or privately;
- ◇ the reliability of the type of car, both when new and older, and its likely running costs;
- ◇ how the value of the car, to purchase or to resell, varies with age;
- ◇ whether to borrow money to finance purchase.

When looking at a particular vehicle, you would note features such as its mileage, colour, condition, etc.

Solution 1.2

(a) The sum is $5^2 + 6^2 + \cdots + 13^2 = \sum_{i=5}^{13} i^2$

(b) Writing out the two sums, we have

$$\begin{aligned}&\sum_{i=1}^{6} 5^i - \sum_{i=1}^{3} 5^i\\&= (5 + 5^2 + 5^3 + 5^4 + 5^5 + 5^6) - (5 + 5^2 + 5^3)\\&= 5^4 + 5^5 + 5^6\\&= \sum_{i=4}^{6} 5^i\end{aligned}$$

(c) The given sum is

$$\begin{aligned}&\sum_{i=1}^{4}(600 + 250i)\\&= (600 + 250 \times 1) + (600 + 250 \times 2)\\&\quad + (600 + 250 \times 3) + (600 + 250 \times 4)\\&= 600 \times 4 + 250(1 + 2 + 3 + 4)\\&= 600 \times 4 + 250 \sum_{i=1}^{4} i.\end{aligned}$$

Solution 1.3

From equation (1.6), the required sum is

$$\sum_{i=1}^{100} i = \tfrac{1}{2} \times 100 \times 101 = 5050.$$

Solution 1.4

We apply the results from Chapter A1.

(a) The recurrence system

$$v_0 = 16\,000, \quad v_{i+1} = 0.85 v_i \quad (i = 0, 1, \ldots, n)$$

has closed-form solution

$$v_i = 16\,000(0.85)^i \quad (i = 0, 1, \ldots, n+1).$$

(b) The recurrence system

$$c_0 = 600, \quad c_{i+1} = c_i + 250 \quad (i = 0, 1, \ldots, n-1)$$

has closed-form solution

$$c_i = 600 + 250i \quad (i = 0, 1, \ldots, n).$$

(c) From part (b), we have

$$\begin{aligned}\sum_{i=1}^{n} c_i &= \sum_{i=1}^{n}(600 + 250i)\\&= 600n + 250\sum_{i=1}^{n} i \quad \text{(equation (1.4))}\\&= 600n + 250 \times \tfrac{1}{2}n(n+1) \quad \text{(equation (1.6))}\\&= 125n^2 + 725n\\&= 25n(5n + 29).\end{aligned}$$

Solution 1.5

We apply the results from Chapter A1.

(a) The recurrence relation

$$v_{i+1} = r v_i \quad (i = 0, 1, \ldots, n)$$

has closed-form solution

[illegible]

(b) The recurrence relation

[illegible]

has closed-form solution

$$c_i = c_0 + di \quad (i = 0, 1, \ldots, n).$$

(c) From part (b), we have

$$\begin{aligned}\sum_{i=1}^{n} c_i &= \sum_{i=1}^{n}(c_0 + di)\\&= c_0 n + d\sum_{i=1}^{n} i \quad \text{(equation (1.4))}\\&= c_0 n + \tfrac{1}{2}dn(n+1) \quad \text{(equation (1.6))}\\&= n\left(c_0 + \tfrac{1}{2}d(n+1)\right).\end{aligned}$$

(d) From the result of part (a), we have

$$v_1 - v_{n+1} = v_0 r - v_0 r^{n+1} = v_0 r(1 - r^n).$$

From the result of part (c), we have

$$\sum_{i=1} c_i = n\left(c_0 + \tfrac{1}{2}d(n+1)\right).$$

Hence equation (1.7) provides the relationship

$$a_n = \frac{v_0 r}{n}(1 - r^n) + c_0 + \tfrac{1}{2}d(n+1) \quad (n = 1, 2, \ldots).$$

Solution 2.1

The long-term behaviour of

$$P_n = (1 + b - c)^n P_0$$

depends on the value of $1 + b - c$, which is positive for positive population sizes.

If $b > c$, then we have $1 + b - c > 1$, and P_n will increase as n increases. This increase goes on forever, becoming more and more rapid.

If $b = c$, then $1 + b - c = 1$, and P_n stays constant at P_0.

If $b < c$, then we have $0 < 1 + b - c < 1$, and the value of P_n will decrease, getting closer and closer to 0 as n increases.

These three possible cases are illustrated in Figure 2.3, following the activity.

Solution 2.2

(a) Take $n = 0$ at the census date in 1790. Then $n = 100$ in 1890, so $P_0 = 4$ and $P_{100} = 63$ (both in millions). According to the exponential model (equation (2.2)), we have $P_n = (1 + r)^n P_0$, so, in particular,

$$P_{100} = (1 + r)^{100} P_0;$$

that is,

$$63 = (1 + r)^{100} \times 4.$$

Solving this equation for r, we have

$$r = \left(\frac{63}{4}\right)^{0.01} - 1 = 0.028 \quad \text{(to 2 s.f.)},$$

which is about 2.8% per year.

(In subsequent discussions of this model we take $r = 0.028$.)

(b) The exponential model for the US population over this period is therefore

$$P_n = 4(1.028)^n.$$

(c) The year 1950 corresponds to $n = 160$, for which the US population is predicted to be

$$P_{160} = 4(1.028)^{160} \simeq 332 \text{ (million)}.$$

(d) To find when the population reaches 500 million, we need to solve for n the equation

$$4(1.028)^n = 500; \quad \text{that is,} \quad (1.028)^n = 125.$$

Applying the natural logarithm function ln to both sides and rearranging, we obtain

$$n = \frac{\ln(125)}{\ln(1.028)} \simeq 174.8.$$

This predicts that the 500 million population level is passed between $n = 174$ and $n = 175$, that is, between the anniversaries of the census date in 1964 and 1965.

Solution 3.1

(a) The proportionate growth rate $R(P)$ at population size P is

$$\begin{aligned} R(P) &= (2.65 - 0.0015P) - 0.4 \\ &= 2.25 - 0.0015P. \end{aligned}$$

Hence the growth for the year which starts at time n is $(2.25 - 0.0015P_n)P_n$. Since this is also the increase from P_n to P_{n+1}, we have the recurrence system

$$P_0 = 8, \quad P_{n+1} - P_n = (2.25 - 0.0015P_n)P_n.$$

(b) The right-hand side of this recurrence relation can be written as

$$2.25P_n\left(1 - \frac{0.0015}{2.25}P_n\right),$$

which is of the logistic form $rP_n(1 - P_n/E)$ with $r = 2.25$ and $E = 2.25/0.0015 = 1500$.

Solution 3.2

The logistic recurrence relation is

$$P_{n+1} - P_n = rP_n\left(1 - \frac{P_n}{E}\right),$$

where in this case $r = 0.028$ from consideration of the proportionate growth rate at low population levels. Since the annual growth rate $P_{n+1} - P_n$ is estimated to be 1.5 (million) for the population size $P_n = 92$ (million), we have

$$1.5 = 0.028 \times 92\left(1 - \frac{92}{E}\right) = 2.576\left(1 - \frac{92}{E}\right).$$

On solving this equation for E, we obtain

$$\frac{92}{E} = 1 - \frac{1.5}{2.576} = \frac{1.076}{2.576}, \quad \text{so} \quad E \simeq 220$$

Hence the prediction, on the basis of the model and of data from 1790–1920, is for an equilibrium US population level of 220 million.

Solution 3.3

(a) The proportionate growth rate is

$$\frac{220 - 100}{100} = 1.2 \quad \text{when } P_0 = 100,$$

and

$$\frac{160 - 400}{400} = -0.6 \quad \text{when } P_0 = 400.$$

(b) Since the proportionate growth rate has the form $r(1 - P/E)$ for a population size P (equation (3.2)), we have

$$1.2 = r\left(1 - \frac{100}{E}\right), \qquad 0.6 = r\left(1 - \frac{400}{E}\right).$$

On dividing through each equation by r and then rearranging, we obtain a pair of simultaneous linear equations in $1/r$ and $1/E$:

$$\frac{1.2}{r} + \frac{100}{E} = 1 \quad \text{and} \quad \frac{0.6}{r} + \frac{400}{E} = 1.$$

Eliminating the $1/E$ term, by subtracting the second equation from four times the first equation, gives

$$\frac{5.4}{r} = 3, \quad \text{that is,} \quad r = 1.8.$$

On substituting this value for r into the first equation, we obtain

$$\frac{1.2}{1.8} + \frac{100}{E} = 1;$$

that is,

$$\frac{100}{E} = 1 - \frac{1.2}{1.8} = \frac{1}{3}.$$

Solving this equation gives $E = 300$.

(c) From part (b), the logistic recurrence relation is

$$P_{n+1} - P_n = 1.8P_n\left(1 - \frac{P_n}{300}\right).$$

If $P_0 = 200$, then we have

$$P_1 = 200 + 1.8 \times 200\left(1 - \frac{200}{300}\right) = 320.$$

Hence if the experiment is started with 200 beetles, then there will be 320 in the next generation, according to the model.

Solution 5.1

(a) The population has $0.3P_n + 8$ 'joiners' each year and $0.4P_n$ 'leavers'. Hence its annual growth rate is $-0.1P_n + 8$, which equals the change from P_n to P_{n+1}; that is,

$$P_{n+1} - P_n = -0.1P_n + 8.$$

With the addition of the starting value, $P_0 = 50$, this is equivalent to the recurrence system

$$P_0 = 50, \quad P_{n+1} = 0.9P_n + 8 \ (n = 0, 1, 2, \ldots),$$

as required.

(b) An equilibrium level c for the population corresponds to a constant sequence $P_n = c$ which satisfies the recurrence relation. If there is such a constant sequence, then we have $c = 0.9c + 8$. Hence $0.1c = 8$, so $c = 80$. The equilibrium level for the population is 80.

Solution 5.2

(a) As n becomes large, $3n$ becomes arbitrarily large, so this is also true of the sequence $2 + 3n$. By the Reciprocal Rule, therefore, the sequence $a_n = 1/(2 + 3n)$ converges, with limit 0; that is, $\lim_{n \to \infty} a_n = 0$.

(b) As n becomes large, n^3 and hence also $5 + n^3$ become arbitrarily large. The sequence $a_n = 5 + n^3$ does not converge.

(c) The quantity $(0.6)^n$ becomes arbitrarily small as n becomes large. This is also true of $20(0.6)^n$, by the Constant Multiple Rule. Hence the sequence $4 + 20(0.6)^n$ becomes close to 4 in the long term. For large n, therefore, $a_n = 100/(4 + 20(0.6)^n)$ becomes close to $100/4 = 25$. In other words, the sequence a_n converges, and $\lim_{n \to \infty} a_n = 25$.

(d) The values given by this formula are (starting with $n = 1$)

$$2, \ 4, \ 2, \ 4, \ 2, \ \ldots.$$

We obtain the value 2 whenever n is odd, and 4 whenever n is even. The values of a_n alternate between 2 and 4, so the sequence does not converge.

(e) The approach used in Example 5.2(c) is helpful here. On dividing the top and bottom of the given expression for a_n by n, we have

$$a_n = \frac{60}{3/n + 5}.$$

Now $3/n$ converges to 0 for large n, so $3/n + 5$ converges to 5. Hence, as n increases, a_n becomes arbitrarily close to $60/5 = 12$. We conclude that a_n converges and that $\lim_{n \to \infty} a_n = 12$.

Solution 5.3

This is an infinite geometric series, with $a = 1$ and $r = -\frac{1}{2}$. Hence, by equation (5.6), the sum is

$$\sum_{i=0}^{\infty}\left(-\tfrac{1}{2}\right)^i = \frac{1}{1 - \left(-\frac{1}{2}\right)} = [illegible]$$

Solution 5.4

Put $s = 0.454\,545\ldots$. The repeating group, '45', is 2 digits long, so multiply s by 10^2, to obtain

$$100s = 45.454\,545\ldots = 45 + s.$$

Hence we have $s = \frac{45}{99} = \frac{5}{11}$.

Solution 1.1

(a) The sum may be written as

$$\tfrac{1}{2} + (\tfrac{1}{2})^2 + \cdots + (\tfrac{1}{2})^{10} = \sum_{i=1}^{10} (\tfrac{1}{2})^i$$

[illegible]

so $m = 0$ and $n = 9$.

(b) By the result of Example 1.1(c), with $a = 1$, $r = \frac{1}{2}$ and $n = 9$, the value of this sum is

[illegible]

$$= 1 - (\tfrac{1}{2})^{10}$$
$$\simeq 0.999\,02.$$

Solution 1.2

(a) The sum of the integers from 51 to 100, inclusive, is

[illegible]

Now, by equation (1.6), we have

$$\sum_{i=1}^{100} i = \tfrac{1}{2} \times 100 \times 101 = 5050,$$

$$\sum_{i=1}^{50} i = \tfrac{1}{2} \times 50 \times 51 = 1275.$$

Thus the given sum is equal to

$$5050 - 1275 = 3775.$$

(b) The numbers given form a finite arithmetic sequence whose terms can be expressed as

$$7 + 10i \quad (i = 51, 52, \ldots, 100).$$

Using equation (1.5), we obtain the sum of these terms:

$$\begin{aligned}\sum_{i=51}^{100} (7 + 10i) &= 7 \times 50 + 10 \sum_{i=51}^{100} i \\ &= 350 + 10 \times 3775 \quad \text{(from (a))} \\ &= 38\,100.\end{aligned}$$

Solution 2.1

(a) Take $n = 0$ at the census date in 1900. Then $n = 50$ in 1950, so $P_0 = 1.65$ and $P_{50} = 2.52$ (both in billions). According to the exponential model (equation (2.2)), we have $P_n = (1 + r)^n P_0$, so, in particular,

$$P_{50} = (1 + r)^{50} P_0;$$

that is,

$$2.52 = (1 + r)^{50} \times 1.65.$$

On solving this equation for r, we obtain

$$r = \left(\frac{2.52}{1.65}\right)^{0.02} - 1 = 0.0085 \quad \text{(to 2 s.f.)}.$$

This is just under 1% per year.

(b) With this value for r, the world population in 2000 (when $n = 100$) is predicted to be

$$P_{100} = 1.65(1.0085)^{100} \simeq 3.85 \text{ (billion)}.$$

Solution 2.2

Take $n = 0$ this time in 1950, so that $n = 50$ in 2000, $P_0 = 2.52$ and $P_{50} = 6.06$ (both in billions). Then, as before, we have $P_{50} = (1 + r)^{50} P_0$, which on this occasion gives

$$6.06 = (1 + r)^{50} \times 2.52.$$

On solving this equation for r, we obtain

$$r = \left(\frac{6.06}{2.52}\right)^{0.02} - 1 = 0.018 \quad \text{(to 2 s.f.)}.$$

This growth rate of almost 2% (for 1950–2000) is about twice as high as that found in the solution to Exercise 2.1(a) for the period 1900–1950.

Solution 3.1

(a) The proportionate growth rate has the form $r(1 - P/E)$ for a population size P (equation (3.2)). Hence (with P measured in billions) we have

$$0.20 = r\left(1 - \frac{3.02}{E}\right), \quad 0.15 = r\left(1 - \frac{5.27}{E}\right),$$

which lead to

$$\frac{0.20}{r} + \frac{3.02}{E} = 1 \quad \text{and} \quad \frac{0.15}{r} + \frac{5.27}{E} = 1.$$

Eliminating the $1/r$ term, by subtracting 3 times the first equation from 4 times the second, gives

$$\frac{21.08 - 9.06}{E} = 1; \quad \text{that is,} \quad E = 12.02.$$

On substituting this value for E into the first equation, we obtain

$$\frac{0.20}{r} + \frac{3.02}{12.02} = 1;$$

that is,

$$\frac{0.20}{r} = 1 - \frac{3.02}{12.02} = \frac{9}{12.02}.$$

Solving this equation gives $r \simeq 0.267$.

(The estimated equilibrium population of the world is therefore about 12 billion, which is roughly twice the level in 2000.)

(b) The logistic recurrence relation in this case is

$$P_{n+1} - P_n = 0.267P_n\left(1 - \frac{P_n}{12.0}\right),$$

where n is measured in decades and P_n in billions. If $P_n = 6.06$ (in 2000), then we have

$$P_{n+1} = 6.06 + 0.267 \times 6.06\left(1 - \frac{6.06}{12.0}\right) \simeq 6.86.$$

Thus, according to the model, the world population in 2010 is predicted to be 6.86 billion.

Solution 3.2

(a) If $Q_n = E - P_n$, then we have $P_n = E - Q_n$ (and $P_{n+1} = E - Q_{n+1}$). Therefore the logistic recurrence relation gives

$$(E - Q_{n+1}) - (E - Q_n)$$
$$= r(E - Q_n)\left(1 - \frac{E - Q_n}{E}\right).$$

Hence

$$Q_n - Q_{n+1} = r(E - Q_n)(Q_n/E);$$

that is,

$$Q_{n+1} - Q_n = -rQ_n(1 - Q_n/E).$$

(b) If Q_n/E is small compared with 1, then the above recurrence relation can be approximated by $Q_{n+1} - Q_n = -rQ_n$, or

$$Q_{n+1} = (1 - r)Q_n,$$

as required.

(c) The recurrence relation in part (b) defines a geometric sequence with closed form

$$Q_n = (1 - r)^n Q_0,$$

where we assume that $Q_0 > 0$ (that is, $P_0 < E$).

(i) If $0 < r < 1$, then $0 < 1 - r < 1$, so $(1 - r)^n$ is positive and decreases towards 0 as n increases. Since Q_n is the amount by which P_n is below E, this predicts that the values of P_n are always less than E, but approach E more and more closely as n increases.

(ii) If $1 < r < 2$, then $-1 < 1 - r < 0$, so $(1 - r)^n$ is alternately positive and negative, decreasing in magnitude towards 0 as n increases. Hence P_n oscillates either side of E, but again tends towards E.

(iii) If $r > 2$, then $1 - r < -1$, so $(1 - r)^n$ oscillates in sign and increases in magnitude as n increases. Correspondingly, Q_n moves further away from 0, and P_n moves further from E.

Solution 5.1

As you saw in Section 4, the convergence or otherwise of a logistic recurrence sequence depends only on the value of the parameter r and, if it does converge, the limit is E. Solutions can be obtained by reference to the table on page 40.

(a) With $r = 1.8$, we have $1 < r < 2$. The sequence converges, to the limit $E = 200$, with values eventually alternating above and below E.

(b) With $r = 2.25 > 2$, the sequence is not convergent. (The long-term behaviour is, in fact, that of a 2-cycle.)

(c) With $r = 0.4 < 1$, the sequence converges, to the limit $E = 1000$, with values always below E.

Solution 5.2

(a) As n becomes large, $(-0.5)^n$ becomes arbitrarily close to 0. Hence $a_n = 4 + (-0.5)^n$ tends to 4. In other words, the sequence a_n converges to the limit 4.

(b) As n becomes large, $1/n$ becomes arbitrarily close to 0 (by the Reciprocal Rule), but n^2 becomes arbitrarily large. Hence $a_n = n^2 + 1/n$ becomes arbitrarily large, so the sequence a_n is not convergent.

(c) The approach used in Example 5.2(c) and Activity 5.2(e) can be applied here. On dividing the top and bottom of the given expression for a_n by n, we have

$$a_n = \frac{16 + 5/n}{2 - 1/n}.$$

Now $5/n$ and $-1/n$ both converge to 0 for large n, so $16 + 5/n$ converges to 16 and $2 - 1/n$ converges to 2. Thus the sequence a_n converges to the limit $16/2 = 8$.

Solution 5.3

(a) Suppose that a constant sequence $x_n = c$ satisfies the given recurrence relation, $x_{n+1} = 2x_n$. This gives $c = 2c$; that is, $c = 0$. Thus the only possible limit value is 0.

The recurrence system defines a geometric sequence. From Chapter A1, Section 3, the closed-form solution is

$$x_n = 2^n x_0 = 3 \times 2^n.$$

As n increases, the values of x_n grow arbitrarily large, so this sequence is not convergent.

(b) Proceeding as in part (a), we obtain the equation $c = 0.2c$ for possible values of c in a constant sequence $x_n = c$. Again, the only possible limit value is 0.

This is a geometric sequence once more, with closed form

$$x_n = (0.2)^n x_0 = 3(0.2)^n.$$

In this case, the values of x_n tend to 0 as n increases, so the sequence x_n converges to the limit 0.

(c) As before, we seek values of c for which the constant sequence $x_n = c$ satisfies the given recurrence relation, $x_{n+1} = 0.3x_n + 140$. This gives $c = 0.3c + 140$; that is, $0.7c = 140$. Thus $c = 200$, so the only possible limit value is 200.

This is a linear recurrence system, so we apply the appropriate formula from Chapter A1, Section 4. This says that the closed-form solution for the linear recurrence system

$$x_0 = a, \quad x_{n+1} = rx_n + d \quad (r \neq 1)$$

is given by

$$x_n = \left(a + \frac{d}{r-1}\right) r^n - \frac{d}{r-1}.$$

In the current case, we have $a = x_0 = 3$, $r = 0.3$ and $d = 140$. Hence the closed-form solution is

$$\begin{aligned} x_n &= \left(3 + \frac{140}{0.3-1}\right)(0.3)^n - \frac{140}{0.3-1} \\ &= -197(0.3)^n + 200. \end{aligned}$$

Now $(0.3)^n$ becomes arbitrarily small as n becomes large so, by the Constant Multiple Rule, the first quantity in the closed form also becomes arbitrarily small. Thus $-197(0.3)^n + 200$ tends to 200 as n increases. In other words, the sequence x_n converges to the limit 200.

Note that in each of parts (b) and (c), the limit found via the closed form *does* agree with the possible limit value established beforehand by seeking a constant sequence.

Solution 5.4

This is an infinite geometric series, with $a = 1$ and $r = \frac{1}{9}$. Hence, by equation (5.6), the sum is

$$\sum_{i=0}^{\infty} \left(\tfrac{1}{9}\right)^i = \frac{1}{1 - \frac{1}{9}} = \tfrac{9}{8}.$$

Solution 5.5

Put $s = 0.513\,513\,513\ldots$. The repeating group, '513', is 3 digits long, so multiply s by 10^3, to obtain

$$1000s = 513.513\,513\,513\ldots = 513 + s.$$

Hence we have

$$s = \frac{513}{999} = \frac{57}{111} = \frac{19}{37}.$$

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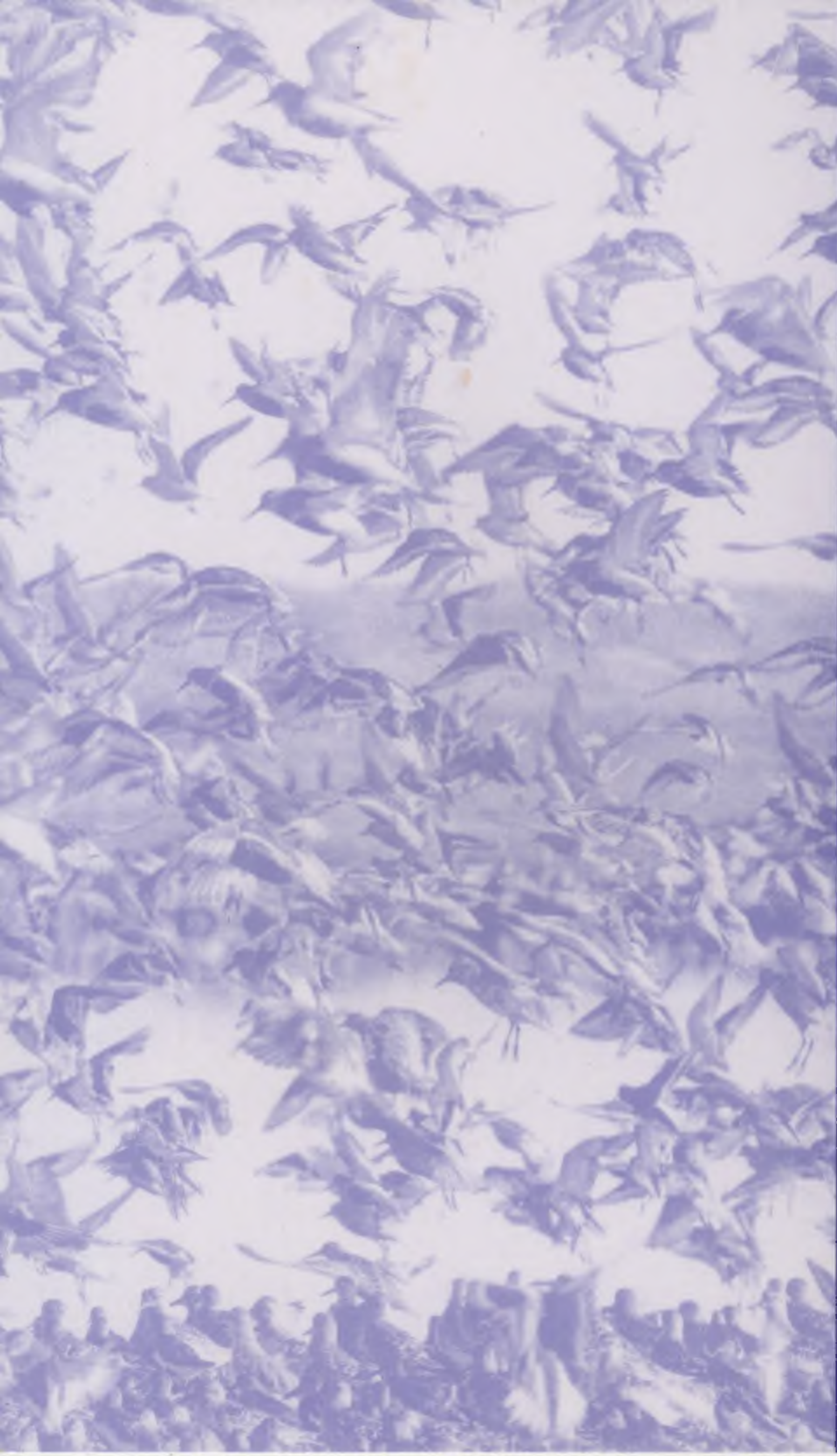

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